On Hilbert's First Problem

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Abstract

The paper shows that the contradiction in Cantor's proof of the nonenumerability of the power set of the natural integers is due to a liar's paradox and not to the nonenumerability of the power set of the integers. The power set of the integers is then enumerated inductively as a graph traversal in an infinite dimensional space. A reductio ad absurdum proof (similar to Cantor's) is given that the set of rational numbers is not enumerable. It is shown in a later section this is also a liar paradox. Next, the continuum is enumerated using prime numbers.

According the foregoing there is a single inductive infinity, and hence the existence of the set of all sets (the universal set which contains all sets except possibly itself) is no longer a contradiction since its power set has the same cardinality. The set of all sets which are not members of themselves can also be inductively constructed via a diagonal construction which pushes the problem off into infinity like other diagonal arguments. Other examples are given. Three other enumerations of the power set are given. It is shown that Cantor's transfinite arithmetic contradicts his own proof that the power set of the integers is not enumerable. The last section shows another proof that the cardinality of the continuum is $\aleph_0$ using a variant of Cantor's transfinite arithmetic.

The section on logic proves that standard propositional logic does not block the liar's paradox which may be imedded in the proofs. Specifically, it is shown that reductio ad absurdum proofs are the source of liar's paradoxes in proofs. Since there are now three sets of proofs; one set consisting of inductive-constructive proofs of the enumeration of the power set of the integers, the rational numbers and the continuum, and the finitistic proofs employing reductio ad absurdum proofs proving that the rational numbers, the power set of the integers and the continuum are not enumerable, we are forced to conclude that either we restrict infinite sets to the *finitistic-contradictive*, so-called "completed infinity" type or to *inductive-constructive* (infinity as a non-halting process) type and that we employ reductio ad absurdum proofs on finitistic infinite sets and not on enumerative inductive-constructive infinite sets. Furthermore, since the reductio ad absurdum proofs contain liar's paradoxes even for finite cases, the use of such proofs should either be banned or propositional logic should be restrictive enough to banish such liar's paradoxes. Finally, the paper makes it obvious that any proof such as the Gödel proofs or any of the others which make extensive use of diagonal contradiction arguments along with inductive constructions are all suspect, especially in the light of the problems with propositional calculus. It's probably possible to push the contradiction in Gödel type proofs off into infinity since it seems that we have a proof of choice; inductively we get one answer and via reductio ad absurdum we get another answer.

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1. Introduction: On the Cardinality of the Power Set of the Natural Integers

The standard proof due to Cantor that the cardinality of the set of integers **Z** is less than the cardinality of the power set (power set) of integers P(**Z**) assumes that $|\mathbf{Z}|=|P(\mathbf{Z})|$, then produces a contradiction and hence "proves" that the assumption that P(**Z**) had the cardinality $\aleph_0$ was the culprit. Of course, an abstraction is usually made in that instead of the integers any equipollent set is used. It will now be shown that this conclusion cannot be drawn from the contradiction. Instead of mapping the set of integers **Z** to its power set P(**Z**), we can change the problem so that S (**Z**) maps to S´ (*any* subset of P(**Z**)) such that the cardinality of S´ is also $\aleph_0$. Since we've changed the problem slightly, we give flexibility to our *demon* (who can do anything in an infinitesimal slice of time); we allow the demon to look into an infinite bucket which holds P(**Z**) and let it select any member of it to place in the set S´. Since we now have a mapping from a set S (say, the positive integers) with cardinality $\aleph_0$, to a subset of its power set S´ such that S´ also has cardinality $\aleph_0$, if there is a contradiction, then it can't be due to the assumption that the sets S and S´ have the same cardinality.

We now make the usual assumptions and redo Cantor's proof. Now define T (as in Cantor's proof) to be a subset of S such that none of its members is a member of its image (in S´). Since T is a subset of S, it will be a member of its power set. However if T is not a member of S´, the demon will find it in the bucket, and then [ceteris paribus] swap it with another set in S´ (whose pair is not in T in S; not that it really matters!) , (we'll keep calling the new set S´ for convenience, only). Since there's a one-to-one correspondence, the set T (in S´) must be paired with an element of S, say s*. Now we ask where we put s* in S. If s* is not in T(in S´), then according to the definition of T we have to put it in T (in S). But then if we put s* in T(in S), then according to the definition of T now s* is in T (in S´) so we can't put s* in T(in S) either. Since S and S´ are equipollent we can't blame this contradiction on the assumption of equal cardinality. We can't escape this paradox by saying that T is empty since if T is empty then when we provide the pairing we have to keep the empty set out of S´ in order not to have a contradiction. But in the case when S´ is P(**Z**) obviously we cannot keep the empty set out of it. If the empty set does show up in S´, then once again it must be matched with an element of S. Since the empty set contains nothing, no matter what it's matched with (say x), then x immediately becomes a candidate for membership in T. But as soon as we put x in T, then we have to put x in T in S´; then this means that we have to take x out of T (in S), and as soon as we do take it out of T, then it immediately means that it must be put in T. So the problem in Cantor's proof lies elsewhere, not with the assumption that Z and P(**Z**) are equinumerous.

We can provide a similar contradiction proof for other infinite sets to prove something which we all know is not true (according to the inductive-constructive definition) to give an intuitive feeling for what's wrong. For example let S be the set of positive integers with some property P; and let S´ be the set of all positive integers. Assume that there's a one-to-one correspondence between these two sets. Now since S is the set only of integers with property P, some of the pairings will be with integers that have P and some that don't have P (call it P´). Now define T to be a subset of S such that none of its members is paired with an integer in S´ with the same property P. So now by the definition of T, the image of T has P´, and the complement of the image of T has P. Since S is a subset of the integers it's also a subset of S´, hence T is also a subset of S´. Now T can't be a subset of its image since its image has P´. It must be in the complement of its image. Because of the one-to-one correspondence, then T (in S´) must be paired with integers in S. Where are these

integers ? Well, nowhere, since by its definition T must be paired with integers having P´ but set S by definition has P. That leaves some of the elements of S´ unmatched hence the cardinality of the integers is greater than the cardinality of a subset of the integers. It would be easy to pick the defining property or predicate P; say S consists of the even integers, then T would consist of even integers matched with odd integers, and hence it cannot be matched with any members in S since S is only the set of even integers.

Providing a characteristic function which merely symbolizes what the words say will not resolve the contradiction. Hence a proof where the characteristic function c(n) is defined as 1 if n (in T) is paired with an odd integer and 0 if with an even , simply uses symbols for words and it's obvious now that this function c(p) will have problems assigning a value to its argument since the "definition" hides the paradox. Since a more general proof of this type, where we have an infinite set of sets instead of simply S, has the same contradiction, it will have an infinite number of characteristic functions which will not be able to assign values to their arguments. The problem is that these characteristic functions are dependent on one another; if one of them changes its value, it affects others. If the demon tried to shuffle the pairings then the values that the characteristic functions assign will keep changing; we'll have a characteristic function chaos in two-dimensions!

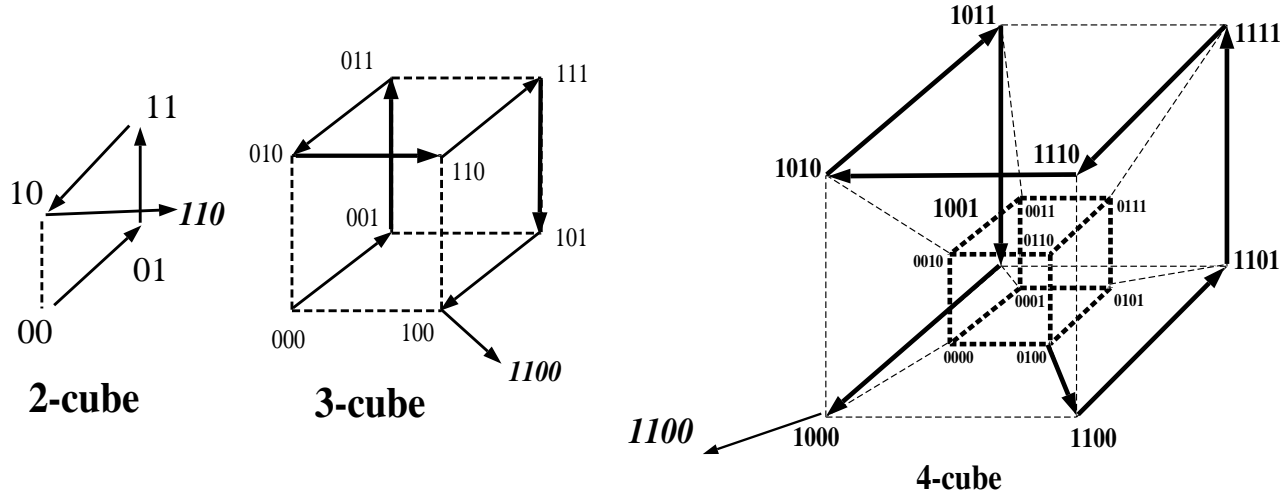
Suppose we try it again to make it even more obvious; this time we'll let S be a small finite set and S´ be a subset of P(S) such that S´ will have exactly the same number of elements. Since this is a finite (and a small set), we can set the demon to work constructing a mapping between these sets and watch it work carrying out Cantor's infinite process (algorithm). Since we've turned the problem into a constructive one we can give the demon all the flexibility it needs by allowing it to pick any of the elements of the power set of S to construct S´ (say from a bucket) as before, however we have to provide a simple condition; the demon must always identify the set T in S and S´ at each step. Suppose the demon starts by selecting any of the elements of S, say z, and picks set from the bucket such that z is not a member of this set. This immediately means that T consists of z. According to the rules, he now has to find {z} in the bucket and put it into S´. Now he has to find an element to put into S that it can pair off with {z}. Say it tries to pair it with some element x in S. Since x is not in the set {z} then the demon duly puts x into T and moves {z} into the T's image. But now, T (in S) is the set {z,x} and thus the demon has to make the appropriate adjustments to T in S´. So it now has to discard {z} from S´ by throwing it back into the bucket and replacing it with {z,x}. But now x is paired with {z,x} and hence cannot be in T. But if the demon does take x out of T, then it has to change T (in S´) again since T (in S) now consists only of z. This starts the loop again. We have the liar paradox. The demon cannot construct the set T even for a finite set.

Suppose the demon tries to outwit the rules by pulling out of the bucket only subsets which contain the element with which they are paired? Then according to the rules, T is the empty set and the demon then must find the empty set in the bucket and put it into S´. But now the element that's paired with the empty set is a candidate for T, and hence the demon is forced to change the composition of T in S´, starting the same paradoxical loop again. Of course, for the finite case, we can relax the rules for the empty set but then if the empty set (which is T) is not in S´ then the paradox disappears since we now don't have to find match for it in S. So the only way to provide a pairing seems to be to provide one in which T is empty and where we allow the empty set not to be a member of S´, but this is impossible in the Cantor's mapping since the empty set is a member of P(S).

Suppose we start as before with S´ empty but this time also with S empty (working again with the integers) and we allow the demon to produce any integer he wants so that he can put it in S. Then as soon as it starts to work, it recognizes that the set T (in S) is empty so he puts {} (the empty set) in S´. Now he can produce any integer x to put in S to pair with {} (in S´). Now since {} by definition is empty and does not contain x, then x by the definition of T must be in T. But if x is in T (in S) then it must be in T (in S´), and hence by definition of T, x cannot be in T! The liar paradox is in the definition of T. There is no way out, since if the demon pairs x with {x}, then T is still empty, and he goes through the same procedure. Now, by induction he can continue this forever without ever having to face the problem of having to pair {} with any integer. We can see the germ of any diagonal proof here; we can inevitably put pairing {} with any integer off into infinity and hence claim that no such problem exists if we use an inductive-constructive approach. However if we consider an infinite set in its totality, then {} must be in S´ and the demon must eventually run into the liar paradox. This theme of the fundamental contradiction between inductive-constructive arguments and impredicative-contradictive arguments that should only be used for finite sets (or finitistic infinite sets) is the fundamental thesis of this paper.

2. Enumeration of the Power Set of the Natural Integers

The basic problem in the enumeration of the power set of **Z** (or any equipollent set) is the fact that it requires higher and higher dimensions instead of the two-dimensional tableaus that are used in enumerating the rationals. However a non-halting algorithm for assigning integers to the P(**Z**) is not difficult to give. Instead of the zigzag path in two dimensions for the enumeration of the rationals we merely need to show a path that meanders through larger and larger dimensions and of course, in the limit, this path needs to traverse an infinite number of dimensions. Consider the algorithm--the word is used in this paper in both infinite and finite senses since we'd have to make up another word for an infinite process if we didn't use the word algorithm-- below for a walk that traverses the vertices of the graphs below.



These n-cubes are graphs of the subsets of sets of size 2,3, and 4. For example, suppose a set consists of $\{x_1,x_2\}$; its subsets are $\{\},\{x_1\},\{x_2\},\{x_1,x_2\}$, which we can write as <00>,<01>,<10> and <11>, so that a zero indicates that the corresponding element is missing from the set whereas the one indicates that the corresponding element is a member of this (sub)set. Enumeration of these subsets can be imagined to take place in two dimensional space; we merely start at the origin (0,0) visit the points (0,1), (1,0) and (1,1). It should be noted that we've used the x_1 and x_2 as both dimensions of the space and as elements of the set merely for convenience and that the ordering of the elements x_i is from right to left (i.e. the higher dimensions are to the left of lower dimensions). The arrows show the walk that visits each node once. We could have written the walk showing dimensions of the walk explicitly by adopting a convention of using a subscript for a *forward walk* (i.e. increasing in that dimension) and a superscript for a *backward walk*. The two-dimensional walk would have been written as $x_1x_2x^1$. It should be noted that the dimension in which the walk is taking place is being counted from the left of the string; thus a transition from (0,1) to (1,1) indicates that the x_1 dimension is not changing but that the x_2 dimension has gone from 0 to 1 indicating a forward walk, whereas the transition from (1,1) to (1,0) indicates that the x_2 dimension is being held constant and that the walk is in the x_1 dimension and is a backward walk.

The algorithm for the traversal in three dimensions is just as easy. We repeat what we did for the 2-dimensional (2-D) walk, then cross over into the next dimension (x_3) and then repeat the 2-D walk but in the opposite direction, so we have the walk $x_1x_2x^1x_3x_1x^2x^1$. All we are really have to do to represent this walk both in zeros and ones, and also to pair each node (i.e. each subset) with an integer is

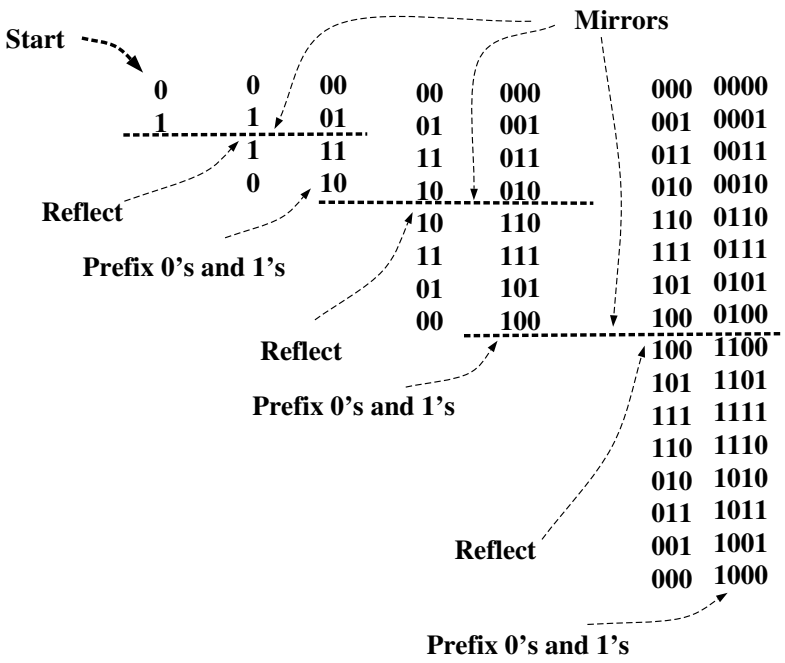
i) start with the string of zeros and ones for the 1-D walk
ii) make a palindrome by repeating this string backwards
iii) prefix the first half (the forward walk nodes) of the digits of the new string with zeros
iv) prefix the second half (backward walk) of the new string with ones.

Note that the crossover into the next dimension gets a prefix of one so that the algorithm splits the subsets into two sets one set of which belongs to the forward walk, the other set to the backward walk. The generalization is obvious; all we have to do is replace the word "1-D" in (i) with "(n-1)-D" and this will give us the enumeration of the subsets of the n-D (i.e. n-cube). For the fourth dimension, we repeat the 3-dimensional walk, cross over into the fourth dimension and then repeat our three-dimensional walk but in the opposite direction exactly as before and we can get the string of zeros and ones for this walk by following the same algorithm as before.

It gets difficult to visualize this walk in higher dimensions however there's no need to do so. The set of strings of zeros and ones produced by this algorithm yields the well-known Grey code also known as the *reflected code*. The reason for the name reflected code is as shown in the figure below.

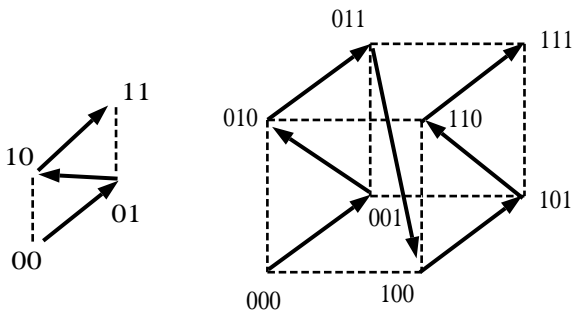
We start off by listing the first two digits as 0 and 1 as shown. Then we imagine that we can reflect this string from a mirror. Thus we get a palindrome of digits. We prefix the *real image* with zeros and the *virtual image* (as in optics) with ones. Now we don't have to imagine any walks in n-dimensional spaces through space warps. Each set of strings of length n produced by this reflection

algorithm corresponds to the subsets of a set of n elements. We now have the algorithm to associate an integer with each element of the the power set of a set of size n, for any n. Indeed now we don't need to visualize anything in any dimension higher than two since as we repeat the reflection algorithm over and over, we'll get a two-dimensional table growing infinitely long in both directions. As the integers continue forever, we can repeat this reflection as long as we want and enumerate the set of subsets of any size. We've made a pairing of the algorithm for the enumeration with the size of the set. Since the power set is enumerable as shown,



it's then relatively easy to pair the subsets of P(**Z**) with the integers in their natural order. Instead of reflecting the binary strings we can merely repeat them and then prefix them with ones. Of course, this corresponds to a zigzag walk as shown in the figure. It hasn't been given for greater than three dimensions because the graph gets cluttered.

Finding the integer mapped to any subset of P(S) is also easy; suppose we want to find the integer mapped to the subset $\{x_{19},x_5,x_1\}$. The binary string and hence the integer corresponding to this set is $2^0+2^4+2^{18}$. In general $\{x_m,x_n,...\}\leftrightarrow 2^{m-1}+2^{n-1}+\ldots$ Of course, the binary string can be represented by polynomials. If we are asked, for example, what number gets assigned to, say, the set of even integers we must of necessity resort to an algorithm. After all, in the pairing of the even integers with the integers, we are given a similar constructive algorithm. The notation $n \leftrightarrow 2n$ is merely shorthand for the construction. Just as it's illegal to ask the question; "What number gets assigned to the largest even integer?", it's illegal to ask the question; "What number gets assigned to the set of even integers?" Such a question belongs to the "infinity as a totality", "infinity as a completed set" mode of thought. Just as in the case of the mapping of the even integers to the integers, where we can only point to the function and say that for any n we can find an even integer 2n, we have to produce the algorithm that will assign a number to any set of even numbers. In this case, there's no problem; the number is



Zigzag Walks

$$2^1+2^3+2^5+2^7+\ldots+2^{n-1}.$$

In case we did not accept the inductive enumeration, we can also claim that the even integers are not

mappable to the integers, and via a contradiction show that the set of integers contains integers which are not in the set of even integers.

Now since most mathematicians believe the continuum hypothesis, then this is enough of a proof that the continuum is enumerable. However there might still be some suspicion that the diagonal proof of Cantor provides a contradiction proof that the continuum is not enumerable and hence that the cardinality of P(**Z**) is less than the cardinality of the continuum. So now we can take a look at the [in]famous diagonal argument to see if it's consistent with other things that Cantor himself proved.

3. Are the Rationals Enumerable?

Consider the rational numbers between zero and one. In their representation in any base :

i) some of these numbers will be of finite length (i.e. a finite number of digits)
ii) some will be of finite period (finite string of digits) and will repeat infinitely many times
iii) some will be of infinite period and will repeat infinitely many times

There is obviously an infinite number of the rationals in any of these sets. The first case needs no discussion. The second case says that there are numbers which repeat and that every repeating string of digits is a rational number. Since any repeating string of digits can be written as a power series which can be summed, the second doesn't need much discussion either and is already known.

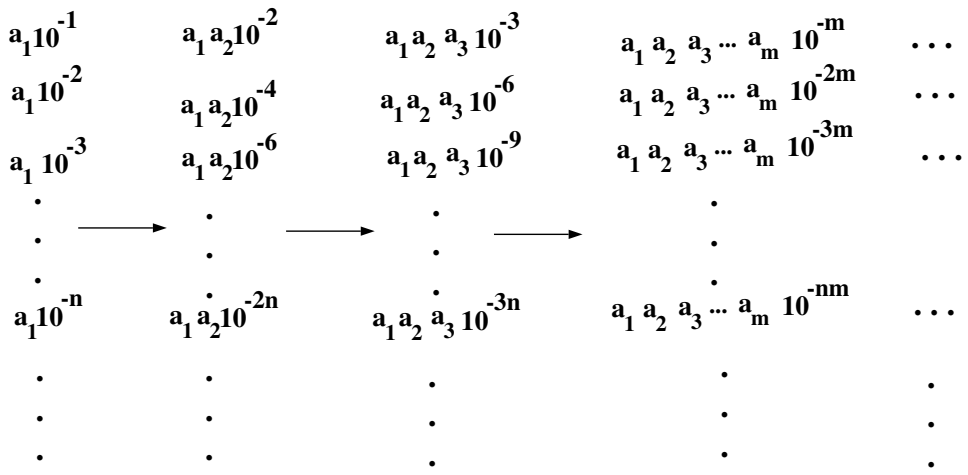
Now we ask the question; "Is there a rational number with the largest finite period?" Let's assume that such a number exists and let its period be L digits. Then all we have to do is to add yet another digit to this period and now we have a rational number with a longer period. Hence there is no rational number with the largest finite period and thus there are an infinite number of rational numbers of infinite period which repeat an infinite number of times. Now that we have an infinite set of numbers which are of infinite length (of digits) and we can pull the Cantor trick. However before we do that we should look in more detail at the infinite period rationals. On the one hand, when we think of something with an infinite period we think of zero frequency (no repetition); on the other hand, the infinite period rationals exist and are infinite in number as shown by the contradiction proof similar to the proof of the infinite number of primes. Since these numbers were derived via summation of repeating finite period rationals they must also repeat and hence must be rational. We have no choice (since it's the harder case) but to accept the rationals with infinite periods with infinitely many repetitions of this infinite sequence of digits. Of course, the arguments against this will also cause contradictions, and are treated in the last section.

We can now proceed with variants of the essentially same proof. One would be to give this last set to the demon so that it can chop off the unnecessary repetitions of the infinite number of infinite periods of the rationals with the infinite periods. Assuming now that these rational numbers between zero and one are enumerable and that the demon has done its job and given us an enumeration of them, we produce the Cantor diagonal number from this set. Since we now have a number which does not match any of the rationals of finite or infinite length (infinite sequence of digits), it's not in

the set of rational numbers. We therefore conclude that the rational numbers between zero and one are not countable! Either the rational numbers between zero and one are enumerable or the Cantor diagonal argument is not a valid proof. Since the rational numbers are countable because we have a constructive proof (by Cantor himself), the Cantor contradiction argument must be the culprit.

Asking the demon to chop of the unnecessary infinite number of repetitions of the infinitely long sequence of digits is not really necessary. We can just as easily pull the diagonal argument on the full length (i.e. infinitely repeating and with infinite period) numbers and deduce that the diagonal number is not in the set. Since we have numbers which have infinite periods and which also repeat infinitely many times, we can produce a diagonal number with the same property; all we have to do is to take the diagonal number and repeat it an infinite number of times to make it rational. Hence we have a rational number which has an infinite period, differs from every rational number in at least one place, and is therefore not in the set of rational numbers between zero and one.

If the idea of asking demon to do more work such as truncating infinitely repeating sequences of infinite periods is not appealing, or if the idea of constructing an infinitely long sequence of digits and then repeating this sequence of digits is not constructive enough, we can directly construct a rational number which will have infinite period and will also repeat an infinite number of times and which is still not in the set of rationals. We start as before with the rationals enumerated conveniently. We produce the first digit of the diagonal number as before but now we treat the first digit of the diagonal number by treating this digit as the term of an infinite power series as shown in the figure. The figure shows the series in base ten but obviously it's only for convenience.



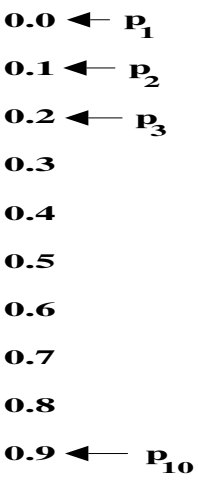
The first column is the diagonal number after picking a_1 to be different than the first digit of the first rational number in the set. Since this number is a power series in base ten as shown it repeats with a period of one. But it clearly differs from the first rational number in the set of rationals and it is also very clearly a rational number. The second digit is also selected to be different from the second digit of the second rational number in the set and we produce the number which is the sum of the terms of the series in the second column. The number $a_1a_2 10^{-2} + a_1a_2 10^{-4} + \ldots + a_1a_2 10^{-2n} + \ldots$ is rational and summable. Once again, pointing out infinite sums of rationals which magically become irrational at infinity is itself as much a contradiction as a rational number with an infinite period (which is usually interpreted to mean that it never repeats) and which also repeats. This is taken up in the last section.

As in the Cantor argument the a_i are merely treated as the digits of the diagonal number, (not multiplied by one another) but in this case they are treated as integers so that multiplying them by the proper power of the base (ten) makes them rational numbers between zero and one. The algorithm is really no more complicated than the Cantor argument; the only addition is that we sum up an infinite series at each step. Since the terms of the series constitute a power series, a finite length operation is needed to produce the diagonal number at each step. Furthermore as can be seen, the diagonal number will always repeat, hence there's no reason for it not to be rational. If we believe in the Cantor diagonal argument we have just proven, once again, that a subset of the rational numbers between zero and one is not enumerable. Since even the cardinality of the repeating integers with finite periods (or even the rational numbers which terminate) is $\aleph_0$, for this proof we didn't even need to consider the infinite number of rational numbers with infinite periods and which also repeat an infinite number of times.

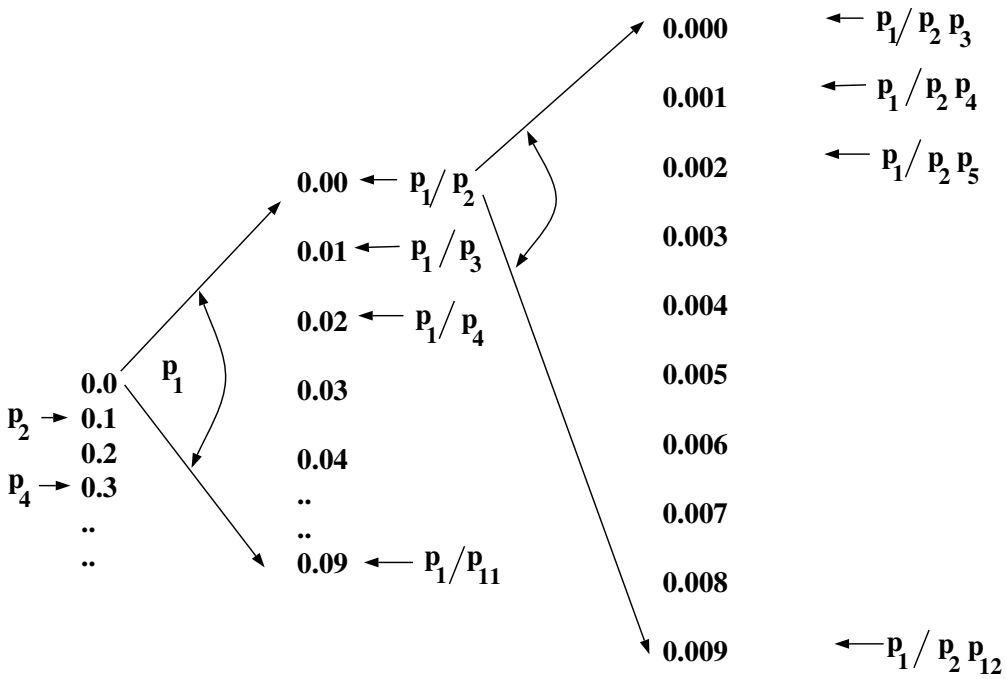
In any case, we now have another contradiction; either the rationals are enumerable (via a construction-induction proof) or there's something wrong with the diagonal argument above which proves that the rationals are not enumerable. We are faced with a paradox similar to the one above for the alleged nonenumerability of the power set of the integers. If we accept the proof above to indicate that the Cantor diagonal argument (for the nonenumerability of the reals) is not valid then the only thing left to do is to imagine the unimaginable and enumerate the continuum.

4. Enumeration of the Continuum

Using the decimal numbering system (any base can be used really) we associate a number with each single digit number between zero and one as shown in the figure, where the p_i are the prime numbers in increasing order. Thus p_1=2, p_2=3, and so on. Now we produce the set of all two-digit real numbers from the above set by first associating a set with each single-digit number. We now associate a number with each of these numbers above by dividing the associated prime number by prime numbers beginning with the next prime as shown below (next page).



Since there are an infinite number of primes, we can continue this ad infinitum. We have the same kind of two dimensional induction to infinity as in the Cantor argument and we'd never run out of prime numbers. The same argument would entail if we used any base instead of base ten. We are merely filling up the continuum by reversing the Cantor middle-thirds deletion algorithm. Instead of taking away the middle thirds we are populating the continuum by continuously adding ten new numbers in every interval. Obviously the same thing can be done in binary or base 3. Will this algorithm produce every number in the continuum? It will produce every infinitely long (repetetive) permutation of digits producible within that particular base system. If there is a number that doesn't show up in this set then where is it?



It looks like the first number is zero (0.00000000...) and it's paired with the rational number p_1/p_2p_3... whose limit is zero. Strangely enough the last number (0.999999999...) is also paired with a rational number which is also zero (i.e. $p_{10}/p_{20}p_{30}$...) in the limit. Because of the specific type of mapping, the largest rational number which gets assigned to the continuum is p_{10} but almost all of the rational numbers in the mapping are concentrated near zero. We could have changed the enumeration by forming the products of primes by dividing unity in the form $1/p_ip_j$... and since we could have started with any prime, we could have enumerated the continuum by the set of rationals concentrated as near to zero as we pleased. The argument that the number of the form $1/p_ip_j$...is zero and not a number is illegal for it violates the very definition of enumeration; the sum of power series and limits are the "completed infinity" results. If it were not so then the mapping $n\rightarrow n!$ is just as illegal as $n\rightarrow 2n$. If it's not illegal then we can also use these numbers in an enumeration of the type 1/n!. The algorithm for the prime numbers is just as valid an algorithm as any of these because although more cumbersome it's still an algorithm that can be used, in principle, in an enumeration. Of course, we should have expected this since Cantor himself provided something similar with his middle-thirds algorithm. It seems that the nonstandard infinitesimals of Robinson are really there!

5. The Diagonal of a Single Dimension!

Define a set S(n) such that it is the set of natural integers less than n. Thus S(0)={}, S(1)={0}, S(2)={0,1}, S(3)={0,1,2} etc. Now as n increases without bound the set S becomes the set of the natural integers. Consider now an integer m defined as a function of n as m(n)=n+1. It's obvious that m(n) is not in set S(n) for any n and hence will not be in the set of natural integers defined this way. Furthermore since we can always define m as m(n)=n+x where x is any positive integer, we now have the result that there exist an infinite number of integers which are not members of the the set of integers!

This is the essence of some of the diagonal arguments. We simply define two sets; a small set S, and a larger set L such that S is a subset of L and so that L contains elements not contained in S. Of course, these definitions of the sets have to be mathematically dependent on one another. Now we simply let the smaller set S increase without bound. Because of the dependent definitions of the sets S and L, there will always be elements of L which are not members of S. Hence one set of diagonal arguments work by pushing something out to infinity and having it get lost out there someplace without ever belonging to the set that we are interested in. The other set seem to have the liar paradox buried in the proof someplace. In some cases, the liar paradox can be exposed very quickly, but in others it seems to require a different approach.

But the real problem is that some of these processes do not seem to violate the definition of an infinite set as is presently used. By its very definition an infinite set can be put into one to one correspondence with a part of itself; the whole can equal the part. All we do in the diagonal arguments is to look out from inside the part of the whole as the part becomes the whole while still remaining a part of the "larger" whole. When the part goes to infinity it seems to become the whole(since it's infinite size) and not the whole since there's still a larger one of which it's a part; but that's a part of the accepted folklore of mathematics derived by the one-to-one school. Diagonal arguments seem to be like aikido, in that the what was once considered a weakness of infinite sets now has been turned into a strength in that this strange behavior can now be turned on itself to prove all sorts of things. It seems that one has a choice as to what is truth in diagonal arguments in which things are pushed out to infinity; one can either prove that something is not in an infinite set or is in the infinite set by a clever definition. For example in the single-dimensional diagonal argument we could have argued thus: Define a set L such that L(n) is the set of integers up to and including n+2. Now as n gets large we have both L and S having the same cardinality and hence are equal. But for any n, m is now trapped in L, hence since L(∞) and S(∞) are equal, m is now in S(∞), but it was never in S(n) for any finite n. We have a simple example of strange behavior at infinity. It seems now we have a real axiom of choice! We can pick what we want to be true. For if we had started out with set L being defined with some predicate, then we would have proven that m was in L and hence S. But we started with induction purposefully to show the contradiction. Hence we can either use induction-construction and accept its truth or accept the contradiction proof. For example, suppose m is not in S. But we know that m is in L. And since S(∞) is equal to (identical with) L(∞), then we have a contradiction, hence m must be in S. The least number for which m then loses its property of not being in S is infinity. If infinity is not a number, then there is no least number for which m loses its property of not being in S, hence by contradiction m must always be in S! Of course, in this case we'd probably want to say that although S(∞) and L(∞) do have the same cardinality, they are not identical sets and hence m is not in S(∞). But in infinite-sets-by-definition we don't always have such obvious and clear choices, as can be seen from the previous proofs regarding the cardinalities of the power set of the integers and the continuum.

The diagonal argument for the real numbers (continuum) is not really much different than the pairing of, say, the even integers with the integers. We know very well that as we pair them one set is increasing faster, but we're not allowed to use rates in set theory. We have the same thing in the diagonal increase, for any length of digits in the diagonal the size of the set of numbers grows faster than the length of the digits. For example, using the binary system the number of numbers which are permutations of the digits grows as 2^n where n is the number of digits of the diagonal number. For any finite n, the diagonal number is always outside the square array from which the diagonal number is constructed, hence as n gets large, the diagonal number just as in the single dimensional case

manages to stay out of the set of numbers in the square array for any n. Hence if we tried the enumerative-inductive or constructive approach to this problem we would have never accepted anything like the diagonal argument since it's obvious that the diagonal number is always just beyond the numbers from which it's constructed. But the lumped contradiction proof starts out with the aggregate infinite set and we are told that things behave strangely at infinity so that this is again a case in which the property gets lost only at infinity. We are merely looking out from inside the two-dimensional array and seeing our diagonal number outside of the set, always out of reach and it seems getting further and further away too as can be seen in the figure. For the decimal system the depth of the array of the numbers grows as 10^n. The only way the array can be forced to be square is if we use a base one system, which looks like an impossibility, unless we use a *1+infinitesimal* base system using proper limiting procedures.

The sleight of hand in the diagonal argument is to imagine we already have the infinite set instead of building up the diagonal number for sequences of increasing size inductively. However, the "paradise" that the method of pushing things to infinity to disappear or to only magically exist only at infinity will probably be very difficult to give up. We are faced with induction which says that the integers (in their natural order) display regularity and the contradiction proofs which always seem to say that the regularity (and everything else) breaks down at infinity. Either infinity doesn't exist and the regularity is there in the number system or there is a real infinity at which irregular things magically happen for no reason. There seem to be two kinds of infinite set theory, the inductive-constructive type and the contradictive type. We either accept inductive-constructive definitions of infinity or we accept only contradiction as the method of proving that something exists or doesn't exist or that something can be accomplished or not accomplished. Since the only real way to think about infinity seems to be via an inductive enumeration, it doesn't seem to serve any purpose to use contradiction proofs to show that what the inductive proofs show to be constructible doesn't exist.

1	**10**	**111**	**1101**
0	00	000	0000
1	01	001	0001
	10	010	0010
	11	011	0011
		100	0100
		101	0101
		110	0110
			0111
			1000
			1001
			1010
			1011
			1100
			1101

The diagonal number produced for each set is above it in bold.

We already know what kinds of trouble we can get into by trying to create infinite (or even finite) sets using the axiom of abstraction schema to specify its properties. For example the mapping from the even integers to all the integers has a contradiction which would have been blocked if we had specified a *real* function. On the other hand, we still see contradiction proofs involving infinite sets in which cheap functions are created such as let f(n) be 1 if n is in S and 0 if n is not in S. This kind of a function will not get rid of the paradox mentioned above, for we can just as easily say let g(n) be a function such that it's 1 if n is mapped to an integer that it has the property P´. It seems that

I) One can always find a contradiction proof which proves the opposite of what an inductive proof proves and vice versa. Enough examples have been given in this paper.

II) Apparently high-powered axiomatic set theory with all of its axioms, and abstractions which uses contradiction as proof is merely an effort to reduce inductive-constructive "infinity as an endless process --an unreachable unutterable number" infinite set theory to finitistic "completed infinity", "infinity as a totality, an object" set theory. But inductive infinite set theory will never be reduced to having the same properties as finite set theory. At best we'll have an infinite number of infinite set theories bridging the chasm between finitistic infinite set theory and inductive-constructive infinite set theory. There is no Cantorian set theory and no non-Cantorian set theory. Cantor and the set theorists freely use both contradiction and induction proofs. It's difficult to tell if any proof to date having anything to do with infinity (including rational approximating sequences for the "irrationals") is correct or at least consistent with other proofs.

III) If one insists on using hybrid methods of proof for infinite sets then an appropriate logic is needed which can handle contradictions. For example we have proofs that the rational numbers, the power set of integers and the continuum are all enumerable and are not enumerable. This is the true axiom of choice; either use the inductive proof or the contradictive proof but not both. If you do use both then a logic which allows things to be both true and false will be necessary.

IV) Since so many proofs in mathematics depend on diagonal proofs it's hard to tell if any of them should be believed. Any diagonal proof that uses both induction and contradiction such as Gödel's proofs and others in computability automatically become suspect. It will probably be possible to construct inductive proofs that say the opposite. This also means that any formal theory that uses induction and then uses contradiction in its proofs is suspect of giving true and false answers hence the inconsistency or the incompleteness theorems.

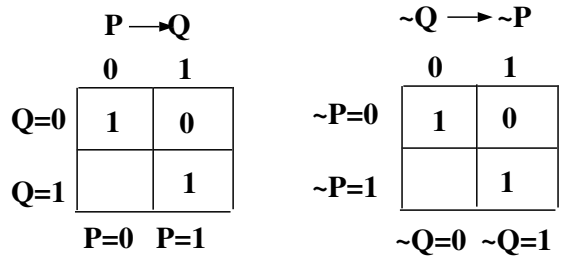
V) And finally, logic (not predicate logic but its roots, propositional logic) itself may be a part of the problem.

6. An Interpretation of Implication, Truth and Falsity

In a discussion of the meaning of implication in logic Rescher [p. 166,1969] states that beginners often experience difficulties and in an effort to explain the assignments of truth values to implication, and that; "It is relatively unproblematic to lead the explanation to the point:" where the table given is shown in the figure. Furthermore he discusses various other possibilities (alternative implications) for the blank spots using quasi-truth-functional systems (for example one due to Reichenbach).

P \ Q	1	0
1	?	0
0	?	?

The purpose of this section is to give a short alternative interpretation for the particular assignments as an introduction to the developments of the later sections. A fuller table is given to the right. The only one we can be sure of is line III. It says that if P=1 and Q=0, then the assertion P→Q (whenever P, then Q) must be false. There are other ways of transcribing the implication into English; they are given in the box below. No matter from which school of logic, no one ever argues that if a counter example can be found for a general statement then the statement is not disproved.



I. 00: P=0, Q=0 ⟵ ~Q=1, ~P=1
II. 01:
III. 10: P=1, Q=0 ⟷ ~Q=1, ~P=0
IV. 11: P=1, Q=1 ⟶ ~Q=0, ~P=0

To justify cases I and IV we should probably make use of modus tollens. If we accept that P→Q is equivalent to ~Q→~P, then the definition of "whenever P, then Q" makes P→Q true for P=1 and Q=1 which by symmetry via the modus tollens implies that the case for P=0 and Q=0 should/could also be assigned the truth value of true.

But we're still stuck with Case II; why should it be assigned the truth value of true? Gamut [p.33, 1991] makes a similar comment that assigning a true value at this point "doesn't seem very attractive to say that it must always be true." However his reasoning is that; "Since we have agreed that indicative sentences are either true or false, let us choose the least unattractive alternative and say that material conditionals are true if their antecedent is untrue." We normally get around this apparent paradox (such as the Hempel Raven Paradox) by defining the implication to mean that there's no causality between P and Q, hence the implication is simply one-sided; that is, Q can occur without P but must occur if P occurs. Of course, there's really no need to repeat the arguments but some of the descriptions are given in the box above. A much simpler and intuitive interpretation is given below for the implication, truth and falsity. On the basis of these simpler definitions some of the number theoretic paradoxes, diagonal arguments, and classical paradoxes will be examined.

7. Ockham's Razor Definition/Interpretation of Implication

If instead of trying to justify all the truth assignments to the implication table, suppose we simply accepted that the only kind of an *absolute proof* is proof of the falsity of a generalization and that the counter example to the general statement is the only acceptable method of proof of falsity. In one stroke we can do away with the objections of the constructivists since the counterexample is a constructive/demonstrative proof. Now that we have a definite definition of falsehood, we can then define truth as not false; that is, if something cannot be proven (via a counter example) to be false then it must be true. However, since we have defined everything not proven (yet) via a counter example to be false as true, then the implication is biased toward truth; that is, the truth value is provisional. Hence we have *definite falsehood* and *provisional truth*. Thus something is true provided that we have not yet been able to prove it (via a counter example) to be definitely false. This interpretation is closer to how we really behave and how science is really done. Science has a bias; like time and entropy it has a preferred direction. Science cannot exist without generalizations; every generalization we make is provisionally true and subject to falsification in the future. Every time a falsification occurs, we shrink the domain of applicability of a scientific theory. This way of looking at the implication also leaves wide open the possibility of multivalent logics since we now don't always have to define everything that is not (definitely) false to be true but can make up other

values such as *unknown, unknowable, indefinite, or meaningless*. We can even have degrees of truth as in probabilistic logic or fuzzy logic. This would also go some way toward easing the *problem of induction* since this definition explicitly states that there is no absolute truth only absolute falseness. Since in theory it would take an infinite number of observations to conclusively decide that all ravens are black or an infinite number of experiments to "prove" that Newton's Laws are valid or an infinite number of additions of integers to prove that x+y=y+x is true, we can leave the truth value to be provisionally true or accept bivalent standard classical logic (SCL) as an approximation of an infinite-valued logic in which the infinitieth value only may be considered to be absolute truth.

8. Some Classical Paradoxes

Liar paradoxes have a long history and are shown in many books on logic. There's a whole book dedicated to the nonstandard set-theoretic disposition of these due to Etchemendy and Barwise[1991]. However, the treatment of these paradoxes in this section will use standard propositional logic and simple (finitistic-axiomatic) set theory. The classic paradox comes from the sentence "All Cretans are liars" uttered by a Cretan which in modern form is shortened to something like "This sentence is false". These cases will be treated separately to show the form of the oscillating or flip-flopping paradoxes. Let L be the set of all sentences which are lies (i.e. false sentences), and C be the set of all sentences uttered by Cretans. Then the sentence S ("All Cretans are liars.") must be an element of the set C since it is uttered by a Cretan. And at the same time, because of the sentence's assertion, set C must be a subset of L, which means that the sentence S also is an element of set L. However, if the sentence S is not true, then it must be the case that C is not a subset of L. Now, since the sentence S is not at element of L, it must be true, and hence all Cretans must be liars, and thus this implies that we should put set C as a subset of L again. In other words we get a flip flopping.

Verificationism, Confirmationism and Falsificationism:

The original idea was that if a theory T implied that an observation O would be obtained and when the experiment was performed and the observation O was obtained, then we'd conclude that the theory T must have been correct. This can be written as;

(I) $\quad O \,\&\, \{T \rightarrow O\} \rightarrow T$

The above is obviously not a tautology so the idea that an experiment proves that a theory is true was dropped and replaced by *confirmationism.* The point of view now is that; the more the predictions come true more we believe that the theory must be true. But standard logic can't handle this. Because after all we'd write this view as

(II) $\quad \{\prod (T \rightarrow O_i) \& O_i\} \rightarrow T$

and because of the definition of $\rightarrow$, it doesn't matter how many correct predictions the theory makes, the above formula never becomes a tautology. On the other hand, *falsifiability* works out. We expect the theory T to yield the observation O, but if instead we get $\neg O$ we conclude that the theory T is incorrect;

(III) $\quad \neg O \,\&\, \{T \rightarrow O\} \rightarrow \neg T$

But of course, that's what implication says; it's true unless it's proven to be false. Since we are using only binary logic and not allowing for anything to be true and false at the same time, and since we see that formal logic is biased in favor of falsity (i.e. if it's false it's definite but if it's true it's only provisionally true) then using this we can say that every experiment is conducted to establish either *definite-falseness* or *provisional-truth*. Thus every experimental set up corresponds to this statement;

(IV) $\quad [O \,\&\, \{T \rightarrow O\} \rightarrow T] + [\neg O \,\&\, \{T \rightarrow O\} \rightarrow \neg T]$

Every observation establishes either that the theory is *definitely-false* or that it is *provisionally-true*. If the predicted observation O is not observed (i.e. when we obtain $\neg O$) then T is *definitely-false*, or if we do obtain the predicted observation O, then the theory is *provisionally-true.*

If we try to depict this situation using propositional logic (since it can easily take care of the all or nothing cases and because we don't need the existential quantifier), then the sentence S says C→L. However since the sentence S was produced by a Cretan we also have S→L; that is, the sentence is a lie. So both of these must be true, thus the first step gives us the statement (S→C)(C→L). Of course since S entails C and C entails L, then S must entail L, i.e. S→L. In other words during this first step in the iteration, we conclude that the sentence itself is a lie. We call this an *iteration* instead of an inferencing sequence since there's something noticeably missing here. In symbols we write this as:

(1) $$(S\rightarrow C)(C\rightarrow L) \rightarrow (S\rightarrow L)$$

which is a tautology. However once we come to conclusion that S belongs to L {i.e. (S→L)} , we then realize that if S is in L (which means that "All Cretans are liars" is a lie) then our conclusion S→L must be false i.e. our conclusion is now (S→L)´. We note that S→L is equivalent to S´+L and (S→L)´ is equivalent to SL´ and we note that by De Morgans laws, these are each others' complements (S´+L)´ = SL´ and (SL´)´ = S´+L. Hence we keep oscillating back and forth between these two values. This is the essence of the paradoxes of this sort.

One can show this more clearly with a paradox of the type in which there's a sentence that says "This sentence is false." If we follow the usual chain(sequence) of inferencing rules, we proceed like this: Let's call this sentence A. Then if A is true, it must be false since that's what it says. Then if it's false, it must be true since the sentence negates itself. At this point we go back to the beginning of the loop. However the statements are supposed to be true for all time. We must collapse *all time* into one sentence. That means our reasoning/inferencing sequence, although itself takes time, must eventually collapse onto a single statement that is true for all time. That is the reason for calling these *iterations* and not inferencing sequences. What we have, if we try to usual sequences is A→Á and from which we derive Á→A. That is we are applying an infinite series of inferences or iterations of the type:

(2a) $A\rightarrow \acute{A}$ Á

(2b) $(A\rightarrow \acute{A})\rightarrow A$ A

(2c) $((A\rightarrow \acute{A})\rightarrow A)\rightarrow \acute{A}$ Á

(2d) $(((A\rightarrow \acute{A})\rightarrow A)\rightarrow \acute{A})\rightarrow A$ A

The results of these iterations have been calculated and are shown next to them. Note that they are all equivalent to A→Á or Á→A. Of course the problem is that we should have never gotten started on this sequence in the first place. The problem is not that the *self-reference* is causing the problem, the problem is not being composed correctly. The sentence is a *compressed* sentence; it's not an atomic sentence but it's a double implication. It states both A→Á and Á→A; if it's true it means A→Á and if it's false it says Á→A. Starting by assuming that it's true is not the whole dimension of the problem. The complete statement corresponds to starting by assuming the sentence is true *an*d also assuming that the sentence is false. These are the two parts of the problem. Both of these must be true at the same time since the statement asserts A→Á if it's true and Á→A if it's false. Obviously, both cannot be true at the same time, and leads to the contradiction which is computed as (A→A´)(A´→A)=(Á+Á)(A+A)=ÁA=0. The statement "This statement is unprovable

[unprovably true]." is obviously exactly the same type of a statement. We can now see that the Cretan paradox is of the same type. It keeps alternating between (S→L) and (S→L)´. This means that both must be true i.e. (S´+L)(S´+L)´= (S´+L)SL´=0. Hence it's a contradiction (as if we didn't already know!).

Russell's famous Barber Paradox as is well-known also oscillates in value. It can be formulated in different ways. In its original formulation, the paradox is set-theoretic. Of course, we know where the paradox comes from. The statements mislead us into believing that the set can be divided into two mutually exclusive sets, H (shaves himself) and B(shaved by the barber). But the fact is that these two sets are not mutually exclusive since the intersection is not zero, i.e. the barber is the intersection of these two sets. The statement that each person is shaved either by the barber or by himself cannot be H+B since this doesn't preclude the possibility of both, so it must be B⊕H. The statement that he who doesn't shave himself is shaved by the barber cannot be H´→B since this doesn't say that the sets are mutually exclusive. What leads us awry is this chain of reasoning; Suppose it's in B, then this implies that it must be in H, which implies in must be in B etc. If we try to write this sequence of reasoning we get

(3a)	B→H	B´ + H
(3b)	(B→H)→B	B
(3c)	((B→H)→B)→H	B´+ H

Off we go into the vicious circle. Neither do things improve if we try to uncompress the sentences differently, as before with the liar paradox;

(4a)	(B→H)→(H→B)	H´+ B
(4b)	((B→H)→(H→B))→(B→H)	B´+ H
(4c)	(((B→H)→(H→B))→(B→H))→(H→B)	H´+ B

The circular reasoning has started again. Of course we can attempt to cut through flip-flopping by making both of these true at the same time. But (B→H)(H→B)=(B´+H)(H´+ B) = B´H´ + BH which is not a contradiction. That still leaves us with a paradox. Let's cast about elsewhere for the source of this by stating the paradox differently. For example none of these are contradictions as simple computations show:

(5a)	(B⊕H)(H´→B)
(5b)	(B⊕H)→(H´→B)
(5c)	(H´→B)→(B⊕H)

However, (B⊕H)(B´H´+BH) is a contradiction since (B´H´+BH)´= (B´H´)´(BH)´=(B+H)(B´+H´) which is equal to B´H+BH´=(B⊕H)´. This boils down to the statement that we should have written the statements as;

(6)	(B⊕H)(B⊕H)´

This is not intuitively clear so let's write it as:

(7) $(B \oplus H)(B \rightarrow H)(H \rightarrow B)$

The first part says that the B and H must be mutually exclusive sets and the second and third parts are the ones in which we try to assign the people in the village to either B or H and get into a loop. We can see this quite clearly since we can calculate

(8) $(B \oplus H)' = (BH' + B'H) = (BH')'(B'H)' = (B'+H)(H'+B) = (B \rightarrow H)(H \rightarrow B)$

The autological-heterological paradox is of this type. The resolution of the barber paradox depended on the fact that the statement did not consider the other two possible cases; *both* (existence of an element of the set of people in the village who belongs to both sets) and *neither* (elements of the set of people in the village who don't shave at all). The resolution of this paradox in predicate logic is merely symbolism; we already know the answer since this is a finite case, and we write it in symbols which express the answer, namely that the barber either shaves or doesn't shave. In this case, we could have also derived a contradiction by using the auxiliary statement $B' H'$ (that there exist objects which neither shave themselves nor are shaved by the barber). When we are dealing with objects (members of infinite sets) which are defined by predicates and if the predicate is highly abstract, it's not so clear that the predicate is falsifiable (or sometimes both falsifiable and provable [to be true]) as in the examples in the paper. Neither can we be sure that we are dealing with objects which can either be put in both (allegedly mutually exclusive) sets, nor can we be so sure that there don't exist objects which belong to neither set.

The real source of the paradox is in propositional logic. We can always deny that such paradoxes exist by calling them merely semantic paradoxes not mathematical paradoxes but it was shown earlier that one of them was hidden inside a mathematical reductio ad absurdum proof. One cannot get rid of this paradox by making up artificial characteristic functions since the characteristic function itself will have problems assigning a value so characterization via functions is not necessarily the answer. Making up such characteristic functions boils down to creation of impredicative sets being considered as a totality or an aggregate and we should always expect paradoxes in such sets. The real serious problem seems to be using contradiction and induction in the same proof. It seems at this point that we should use one or the other but not both, then we might expect two kinds of infinite set theory; finitistic *impredicative-contradictive* and *inductive-constructive*. The purpose of this long-winded discussion of the barber paradox was to show that the liar paradox may be a set of daisy-chained (inductive instead of recursive) inferences and buried too deeply in the inferencing sequence to allow one to spot it.

9. The Continuum Paradox and Proof via Contradiction

If we examine the Cantor proof for the nonenumerability of the continuum (and the similar proof given earlier for the nonenumerability of the rational numbers) it's not difficult at all to spot the liar paradox. We assume the the continuum is enumerable E, and prove E' by producing a diagonal

number which is not a member of the set. Since a similar proof can be constructed for the rational numbers it's clear now that the proof is a contradiction. We could have spotted this very quickly, if we did not halt our proof as soon as we derived the contradiction. After all, the only thing that was produced was one single number that wasn't in the set of the reals. Now all we have to do is to add this diagonal number to the set of reals and now it's enumerable. But we can pull the diagonal trick again so it becomes nonenumerable again. But as in the analysis of the paradoxes above, we have merely produced E, E´, E, E´, E... so that we should have spotted this as the liar paradox.

The question now is; when is reductio ad absurdum a valid proof? In this case, it was easy to spot but in the case of the nonenumerability of the power set of the integers, it was hidden too deeply. We can look into the nature of contradiction proofs by examining their nature more closely. Suppose we assume E and via a contradiction "prove" E´. We choose E to disprove because we must intuitively feel that E must be false. (Of course, we can do it vice versa). Now suppose we assume EC (E and C) where C can be something that has absolutely nothing to do with the problem, say, something like "Unicorns exist!" Since C has nothing to do with a mathematical proof, obviously, we'll be able to run through the same proof via contradiction and hence prove EC=0. Now this says that one of E or C (or both) is false not necessarily only E.

Case I: Standard logic

Since we've already proven that E=0 we obviously don't see a contradiction here. Since we all accept this as quite natural then we can also assume EC´ and hence "prove" this time that EC´=0. Obviously we can then rewrite what we've done as EC+EC´=E(C+C´)=0 and since C+C´ always true we once again prove that E=0.

Case II: Intuitionism

But now the intuitionist enters; we did not analyze the problem correctly for since EC=0 is equivalent to E´+ C´=1 and since EC´=0 is equivalent to E´+C=1 what we've done is really equivalent to (E´+ C´) + (E´+C)= E´+ (C+C´)=1. Now, it doesn't matter what the value of E is since the expression is always true, so we haven't proven E´ via contradiction according to this point of view.

Of course, we really have done it incorrectly. Let's reconsider.

Case Ia: Standard logic again

What we proved was EC=0 <u>*AND*</u> EC´=0. Hence we actually proved (EC)(EC´)=ECC´=0 so we really have no reason now to claim that we proved E=0. Now we have a contradiction, we proved that E=0 and E=? This is not inconsistent according to the standard definition since we did not prove E and E´ but there's no reason why we should not get the same answer both times, especially since by our original definition, C has *nothing* to do with the problem. But how do we know that C might have been an important part of the proof.

Case IIa: Intuitionism

Let's try it with intuitionism: (E´+C´)(E´+C)=E´+E´C+C´E´+CC´=E´+ E´(C+C´)=E´+1=1 which still says that E is unknown. Since intuitionism would have rejected the original contradiction proof also as not valid, it seems that it gives consistent answers.

The problem is that standard propositional logic doesn't block the liar's paradox. Throwing it out as a semantic paradox is not the answer. The reason we got different answers in the first place with standard logic is that the liar's paradox is lurking inside the proof. We could dodge the inconsistency by saying that since C has nothing to with the problem, it doesn't matter that we get a solution that says E=? and that this is merely another semantic paradox. But it isn't. What if C might have had something to do with the problem at hand and C was the actual cause of the contradiction not the assumption E? This is exactly what happened in the "proof" of the nonenumerability of the power set of the integers. It was shown for the finite case that when T was the empty set, it could not be paired with an integer in S. Suppose we suspected that this was the case in the infinite set enumeration, then we might have concluded that P(**Z**) is not enumerable because T = { } or that the set is enumerable iff T in S´ can be paired with an integer in S. But P(**Z**) is enumerable and { } is paired with zero, so the problem is really, firstly with the impredicative definition of T, and secondly with the nature of proofs via contradiction.

The reductio ad absurdum proof says that $(A\rightarrow A')\rightarrow A$ (or $(A'\rightarrow A)\rightarrow A'$). Let's consider the first case; since $A\rightarrow A'$ computes to A´ what we really have now is $A'\rightarrow A$. Since we already have $A\rightarrow A'$, together they constitute a liar's paradox, a contradiction. However since we have "assumed" A and haven't proven it, the liar paradox is not obvious especially since it could be buried in the proof. The classical liar's paradox is obvious, after all, we assume A then derive A´ immediately but we do not conclude A since it's too obvious that we could have just as easily assumed A´ and immediately proven A´ via contradiction. We could not do the same thing with Cantor proof of the nonenumerability of the continuum (i.e. Assume E derive E´ and assume E´ and derive E) however we could have successively obtained E, E´, E, E´, E .... Nevertheless we now have a constructive proof that it's enumerable, so now we have a real contradiction not a semantic one. So when we obtained the oscillation in truth values we should have concluded that the proof itself was a contradiction and hence not a proof. Therefore the liar's paradox is a trivial case of the alternation in truth values. It would be safer to banish the reductio ad absurdum proof unless we can find conditions under which it can be safely and correctly applied. Of course, this hold true only if we want to abolish reductio ad absurdum proofs for infinite sets which behave like finite sets.

In the case of the Cantor proof of the nonenumerability of P(**Z**), the contradiction was not due to the assumption that P(**Z**) was enumerable but in a part of the proof, but this was too deeply buried in the proof and it wasn't obvious that it had anything to do with the problem. Hence the only security of avoiding such paradoxes would seem to consist in an axiomatization of propositional logic which blocks proof via reductio ad absurdum; after all the proof is right in front of us that we cannot spot a liar's paradox hidden in such a proof.

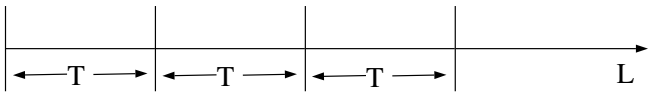
An alternative in using reductio ad absurdum would be not only assume A and derive a contradiction but also try it again by assuming A´. If we can prove both A and A´, obviously it's a contradiction. However, it's not always possible, hence the objection to the proof via contradiction by the intuitionists. Another attempt might be to make assumptions in pairs, such as AB, AB´, A´B, A´B and make an exhaustive analysis. However this also has its problems; the first being to find an appropriate co-assumption B, of which there might be an infinite number thus we could have an undecidable proposition. On the one hand we have an inconsistent propositional calculus (allowing

reductio ad absurdum) and on the other a possibly infinite procedure which implies undecidability. Is it the case that blame is being put on predicate logic whereas in reality the problem is already in propositional calculus? Can any axiomatization of propositional logic that doesn't block the derivation of the liar paradox (which can be hidden in proof via contradiction) really be called consistent?

10. Other Odds and Ends

The universal set **U** is by definition the set of all sets and hence it is the largest set. By this definition, then P(**U**) is a subset of **U**, so that $|P(\mathbf{U})| \leq |\mathbf{U}|$. But by the Cantor Theorem that the cardinality of a set is less than the cardinality of its power set, we obtain $|\mathbf{U}| < |P(\mathbf{U})|$. Now since the Schröder-Bernstein Theorem asserts that if $|\mathbf{Y}| \leq |\mathbf{X}|$ and $|\mathbf{X}| \leq |\mathbf{Y}|$, then $|\mathbf{X}| = |\mathbf{Y}|$ we have (had!) a contradiction that $|\mathbf{U}| = |P(\mathbf{U})|$ and $|\mathbf{U}| < |P(\mathbf{U})|$. But by the results of this paper $\mathbf{c} = |\mathbf{Z}| = \aleph_0 = P(\aleph_0) = |P(\mathbf{Z})| = \infty$, so we have $|\mathbf{U}| = |P(\mathbf{U})|$, if **U** exists (depending on one's preferred set axioms.)

But what is the cardinality of **U** (possibly excluding itself) ? We have the result that there exist rational numbers whose period is infinite and which also repeat an infinite number of times. Since we can never obtain the last digit of the first period, we will never know if a given number never repeats or starts repeating at infinity. We can sketch how these number behave from the behavior of the rationals with finite periods as shown in the figure.



Since the total length (of the digits) is equal to its period multiplied by its frequency of occurrence both of which are infinite we have L = Tf which in the infinite case corresponds to $\infty = T \bullet \infty$ (where we've used the standard symbol as a shorthand for cardinality) which is standard fare in transfinite arithmetic. However in the case of these specific rationals we also have the period T increasing without bound so that we have $\infty = \infty \bullet \infty = \infty^2$, which is also a standard derivation in arithmetic dealing with transfinite cardinals. All we have to do is to iterate this definition to obtain;

(9) $$\infty^{\infty} = \infty$$

There doesn't seem to be anything bigger since we have, applying the same equality

(10) $$\infty^{\infty^{\infty}} = (\infty^{\infty})^{\infty} = \infty^{(\infty^{\infty})} = \infty^{\infty} = \infty$$

There's nothing bigger hence the universal set also has the same cardinality and so does it's power set. A concept essentially defined as never-halting and which has no end cannot be less than or greater than another never-halting process without an end.

The Russell paradox, the set of all sets **Q** which are not members of themselves arises from the same fundamental incompatibility between the contructive-inductive and the finitistic impredicative and contradictive approaches to infinite sets. We can let the demon construct this set; suppose the set of all sets which do not contain themselves $X_1,X_2...X_n...$ are given in some order, say according to their cardinality. Since **Q** is by definition the set of all these sets it will not be the predecessor of any X_n, and no X_n will be its successor. Then the demon can start putting these sets, the X_n, into the sack (i.e. the container of these sets). When (and if) the demon ever finishes this container will be **Q**. However, the demon will never finish, exactly in the same way as he never will be able to enumerate the integers. So we have a diagonal argument (construction) in which the question of whether **Q** belongs to itself is pushed off to infinity; hence the demon will never really have to decide whether to include **Q** in **Q**. On the other hand, if we use the infinite set as a totality, which can be imagined as the demon finishing the job in some finite period of time (this is possible since any finite segment of time has an infinite number of infinitesimal ticks of the infinitely fast demon clock), then we can ask the question of where **Q** belongs. But this is the "completed infinity" point of view.

If we want infinite sets to behave like finite sets then this approach is like the other contradiction proofs; it's valid within its domain. However, then in this domain, we can use contradiction proofs to show that the cardinality of the even(odd) integers is smaller than that of the integers; the cardinality of the integers divisible by three smaller than that; those divisible by four even smaller and so on. Then we wind up with an infinite number of transfinite cardinal numbers less than $\aleph_0$, as well as an infinite number of transfinites greater than $\aleph_0$. Furthermore there is no least transfinite cardinal and no greatest transfinite cardinal. We've merely reproduced the number system where we might arbitrarily pick $\aleph_0$ (or some other aleph) as a convenient "middle" of all the alephs. It has no beginning (like the negative integers which stretch out to infinity) and no end. In the axiomatization of the number system, at least we can use a Hilbert Hotel like argument and pick any integer and call it zero or else we are claiming to be able to find the middle of infinity! In this case, there is no least aleph, and no greatest aleph. We still have the contradiction of the universal set (by definition the largest) being smaller than its power set because of Cantor's proof. We still need to calculate the cardinality of the set of alephs. And we can just as easily ask what the cardinality of the prime numbers is in this scheme of things.

In this case, we'll probably need an infinite number of infinite set theories and probably with an infinite number of axioms to bridge the abyss between the two opposite ends of the spectrum. The situation for the completed infinity, infinity as an aggregate or the finitistic infinity school of thought is actually much worse than seems as can be shown below.

11. Another Enumeration of the Power Set of the Integers

There are two criticisms of the enumeration of the powerset of the integers given previously:

a) The inductive construction only enumerates the finite sets and not the infinite sets
b) What integer gets assigned to, say, the even integers?

Both of these are related and are apparently problems for the simple reason that the *completed infinity* concept is bring brought in to contradict the *inductive infinity* perception of infinity. The very definition of enumeration is inductive, and the idea of the completed infinity can also kill this concept very easily. For example we can ask the questions: When do the even integers become infinite? When do the odd integers become infinite? Indeed when do any of the arbitrary infinite subsets of the integers become infinite? Do they hang separately or together?

A mystic belonging to some religious order stops at a village during his travels. Impressed by the mystic's aura and his apparent mastery of the unfathomable secrets of the universe the peasants gather around to avail themselves of his beneficence. After being wined and dined, the mystic asks the villagers if they have any problems he can tend to. One of the villagers complains about the summer heat. "Well, then" says the mystic "get me a gigantic rope, long enough to go around the mountain and I'll move it away so that it doesn't block the cool breeze from the ocean."

Completely dazed by the evidently supernatural power of the mystic, the villagers, after a discussion among themselves, tell the mystic that they can have such a rope by next year. The mystic shows up the next summer, and asks the peasants to draw the rope around the mountain. After weeks of huffing and puffing the villagers manage the task. The mystic grabs hold of the two ends of the rope:

"Ok, now, lift it up and put it on my back!" The villagers are startled. "How can we do that?" they ask. "Well" says the mystic, "if you don't put it on my back, then how can I take it away?"

The fact that we can write mapping of the even integers to the integers simply as $n \rightarrow 2n$ is a mere artifact of symbolism. We can also write the mapping $n \rightarrow n!$ in very simple notation. All these symbolisms mean is that for any n we can find 2n or n!. We are never allowed to ask the question of what is the number n! when n becomes infinite; the very question is illegal. Of course, this is the inductive infinity definition and we can use it when we want it; we just don't allow the completed infinity idea to mess up our proofs when we don't want it to. However, when we need it we can merely put all the integers in a sack, and then draw circles on the blackboard (or the textbook) so that we can tell our students that everything is inside the circle just like Ragu sauce! However, the very definition of the power set puts the Catch-22 to work. As finitist/infinititists we now have a right to demand that the numbers, which get assigned to an infinite number of infinite sets which are all subsets of the powerset, be produced. And the Catch-22 doesn't allow us to invoke the inductive construction that, in principle we can produce the number, (which was given in section 2). But this Catch-22, which is merely a problem of formalism, now allows us to make a statement that the cardinality (which is really another sophistical name for size or magnitude) is greater than that of the integers. Of course, we cannot now protest that something is wrong, for n! is greater than 2^n for all $n > 4$. Induction has already been ruled out via the first Catch-22. And even if it weren't we have yet a more powerful Catch-22 lurking around the corner; it says that "Even if n! is greater than 2^n for any finite $n > 4$, it doesn't mean that it'll always be greater, because at infinity anything can happen. Infinity is a strange beast. For example, sums of rational fractions add up to an irrational number. As another example, our intuition fails in the case of the cardinality of the, say, even integers because we think it's smaller but actually it has the same cardinality as the integers!" The usual slip-flop between the two modes whenever it suits whoever wants to prove whatever he wants to prove seems to be rule of the glass bead game of the century.

Sorites Paradoxes:

If a man has a penny, he is not a wealthy man.
If a man is not a wealthy man, then giving him one more penny will not make him a wealthy man.
Therefore, regardless of how many pennies you give a man, they will not make him a wealthy man.

The integer 1 is finite.
If the integer n is finite, then n+1 is finite.
Every integer is finite. The integers by definition enumerate themselves. Therefore the set of integers is finite.

First of all, the the power series summation is another example of the completed infinity concept and it's at odds with the inductive infinity. The inductive infinity says that any number is infinitely divisible. Zeno's paradox is a case in point. The infinite divisibility of any number, rational or irrational number is another one. The infinitude of an infinite set of integers is another manifestation of the same phenomenon. The fact that we can only sum infinite fractions with some regularity (i.e. the common power series) is only a testimonial to the state of the incompleteness of our mathematics. The fact that we can call the infinite fraction 0.99999999.... unity is only another use of Alexander's Sword to shortcircuit the inductive infinity to become a completed infinity. Since any number is infinitely divisible if we insisted on the inductive infinity we can refuse to mix inductive and completed infinity concepts. We accept the summation of the power seris because it's obviously useful. Now, this useful solution of the infinity carries over into the integers, the power set and the continuum so now we ask for completed infinities when we want them and insist on induction when we're cornered. We could have just as easily insisted on an explicit enumeration of any infinite subset of the integers instead of integers which can be described by a neat and clean notation such as the even integers. It should be possible since any infinite subset of the integers has the same cardinality as that of the integers. But this gets conveniently shorcircuited, since by definition the integers can be enumerated.

"Son", says the father, " you see double, because you're cross eyed."

" That cannot be, father", says the boy "if what you say were true then I'd see four moons instead of only two."

Ornstein (Psychology of Consciousness)

However, in the discussion of infinite sets and power sets arguments brought over from this kind of completed infinity will obviously bias the results in a particular direction. The very inductive concept of infinity as being endless process makes it difficult to imagine it being less than or greater than any other endless process. If we want to shortcircuit this and produce a complete infinity then the very idea of cardinality of infinite sets and enumerability is no longer what we imagine it to be; it's merely a reflection of conflicting definitions and incompatible notions of infinity.

Keeping in mind the two dichotomies concerning infinity, we can produce a number for the subsets of the powerset: Some numbers are given below, where $\rho = 2$ because we've used the binary system in section 2:

11a) Even integers : $\sigma_2 = \rho^1 + \rho^3 + \ldots + \rho^{2n-1} = (\rho - \rho^{2n+1})/(1-\rho^2)$

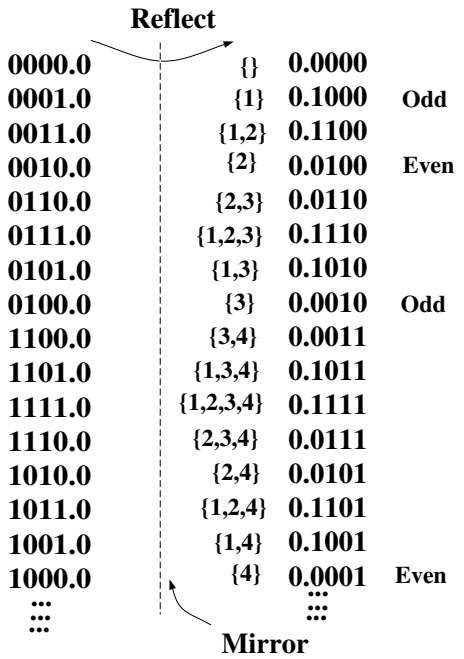
11b) Integers divisible by 3: $\sigma_3 = \rho^2 + \rho^5 + \ldots + \rho^{3n-1} = (\rho^2 - \rho^{3n+2})/(1-\rho^3)$

Of course, we can produce such series for any set defined by some regularity. The fact of the matter is, it doesn't make any difference how large they get in the simple linear inductive-enumerative

process, since that's its very definition. So we can hide behind the symbolism n $\rightarrow$2n, n$\rightarrow$2n+1 or even n$\rightarrow$n! for to deny it would be to deny the whole concept of cardinal numbers, cardinality of infinite sets, and threaten the whole concept of the completed infinity, infinity as a totality or an object, and with it much of the foundations of mathematics. But now by the very definition of the concept of the power set, and the impredicative definition "completed infinity" concept of sets, a sibling of the "completed infinity" gets us with its own Catch-22's:

i) Prove that it's a number!
ii) Give me the number, for example, for the set of integers which are not divisible by three!

We can't escape (i) by pulling the induction argument that it will produce a number for any n, since the completed infinity demands a number, whereas even for the mapping n$\rightarrow$n!, the defenders of the status quo can take refuge in inductive enumeration even if n! grows faster than 2^n. It seems as if the number of Catch-22's is complete and remorseless; we've been put in the ring with our arms tied behind our backs and we can't complain. It looks like the definition of cardinality is merely a function of our inability to deal with nonlinearity and has nothing to do with what we'd normally consider to be some proxy for magnitude.



So we merely let ρ=1/2 so that the series becomes a summable power series and hence a rational number. All we have to do is affix a radix point and reflect the bitstrings as shown in the figure . But the second challenge (ii) by its very impredicativeness means that we cannot produce such a number at will, for there are many such infinite sets in the power set. We don't have any escape clauses except to use the same sophistry; we merely ask that *we be given the set*, and we'll give the inductive construction recipe for the number. There is no other way to do this except to put the burden back on the questioner. Peano's Fifth Axiom is just such a completed finitistic infinity; someone has frozen time for an infinity, has collected all the infinite number of integers and has put it in an infinite sack for us (and restarted our time), such that if we look there and find any integer n, its sucessor n+1 is guaranteed to be there so we need not search. It merely cleverly hides the infinite process and in fact, pushes off the problem into infinity and also not infinity since every number is finite, and we don't have to look around for any number since no matter how large it is, it's guaranteed to be instantaneously there, and yet at the same time the set contains all of the infinite number of integers in it! A true miracle, since by definition the integers enumerate themselves, and they're all finite!

Of course, it would be simple to merely imagine that such an impredicative set could, in principle be constructed; mathematicians were apparently the first people who conducted Gedanken experiments. If such a set can be given to us (completed infinity), then the recipe for constructing the number for this set, is just as easy; just look up the positions of integers in their natural order and construct the series (in any base really).

At this point, of course, the cousin of the first two Catch-22's shows up. "Since the resulting series is irregular (not a power series) and continues forever (inductive infinity mode now!) it can't be summed up, hence the number is irrational." The recipe for constructing repeating fractions from any arbitrary list of digits was already given earlier; all we have to do is to repeat it.

But now, the long lost cousin of these Catch-22's, Sophistry shows up. "It is true that there is no rational number with the longest finite period but that doesn't mean that its period is infinite; it just means that the period is *arbitrarily large*." Of course, anyone can perform a Gedanken experiment, not only the completed infinititists. Let's imagine we have two printers, $\mathcal{N}$ and $\mathcal{R}$ connected to a supercomputer. The printer $\mathcal{N}$ is programmed so that it never prints decimal points. Now we program the supercomputer to send the digits of π to both printers; do the digits printed on $\mathcal{R}$ become infinitely long and the digits printed on $\mathcal{N}$ merely *arbitrarily large*? So far it looks like we have to drive a car with only a single forward gear, no reverse gear, and which can only make left turns. Before you can go anywhere, you have to chart a careful course to avoid getting stuck at intersections forever or being forced by one-ways along the way to go somewhere completely unintended. Nobody would buy such a car. There seem to be roadblocks set up everywhere to foil the would be traveler anyway, who would venture out with such a car?

In any case, we do have a cookbook recipe (an unhalting one of course) that will construct a rational number for any of the subsets of the power set. The construction of the numbers for the finite subsets of the powerset obviously is possible since there are only a finite number of them and we can always sum a finite series. In fact, we don't even have to make these numbers rational fractions at all, we can just as easily leave their numbers as integers. Since both the rationals and the integers have the same cardinality $\aleph_0$ and one of the cardinal rules of transfinite arithmetic is that $\aleph_0 = \aleph_0 + \aleph_0$, the numbers are still enumerable. Even if we had left the enumeration with the fractions the enumeration would have been complete and legal.

12. Yet Another Pedestrian Enumeration of the Power Set of the Integers

There is yet another way we can enumerate it. For any finite set it's very easy. All we have to do is to take the integers in the set and produce a rational number from them by using any of the permutations of the digits and by sticking a decimal point in the front. For example, for the set {1,3,9} we have the choices 0.139, 0.193, 0.319, 0.391, 0.931, 0.913. For sets with larger integers such as {11,101,1111} we may do one of two things; we can either follow the same algorithm as above or we can fix the problem so that the possibility of any string of digits in the rational number recurring will be zero. Find the largest number in the set--(in this case it's 1111)-- pick a base for the rational number system which is one larger than the largest number in the set; in this case the base of the numbering system will be 1112. In this base system each of the integers in the set will be a unique symbol, say $\{b_{11},b_{101},b_{1111}\}$ so that we merely construct the rational number for this set as $0.b_{11}b_{101}b_{1111}$. Clearly every such finite set will always yield a nonrepeating rational number in some base system. Since we can always find a larger integer for any integer ever given in any set we can always represent any set of integers as a rational number in some ever increasing base system. Now, by induction we'd like to say that since this will always be possible if we continue this forever (inductive infinity again), we can definitely enumerate the power set of the integers. But once again, the Catch-22 virus spoils us, we cannot use induction in this case, because that's the function of the

Catch-22; to spoil induction (which it uses very conveniently when it suits its purposes) and force a number to be constructed and given (completed infinity mode again). Can we produce such a number? We have to resort to the same sophistry; yes, give me the numbers in the infinite set, and I can construct the rational number for it. Suppose the specific infinite set consists of $\{a_1,a_2,a_3....a_m...\}$ the rational number for this set is given by:

(12a) $$a_1a_2a_3...a_m10^{-m} + a_1a_2a_3...a_m10^{-2m} + a_1a_2a_3...a_m10^{-3m} +\ldots + a_1a_2a_3...a_m10^{-nm} \ldots$$

which is similar to Cantor's proof that the unit square has the same cardinality as the continuum. The number in decimal form would be;

(12b) $$0.\ a_1a_2a_3...a_{m...}\ a_1a_2a_3...a_m \ldots a_1a_2a_3...a_m \ldots a_1a_2a_3...a_m \ldots$$

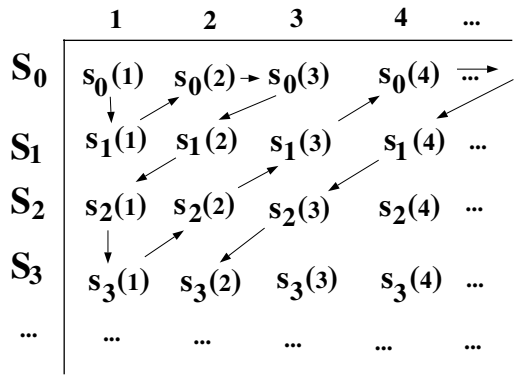
Clearly it repeats just like a rational number should, and its period will be infinitely long (or is it merely *arbitrarily large* ?) and it will also repeat an infinite number of times. At this point we even get a chance to present a question to the Catch-22 viruses; "Is the string of digits of any of the infinite subsets of the power set of the integers infinite or is it merely *arbitrarily large?* If it's arbitrarily large (and if this phrase is not mere sophistry but actually means something) then is there any point to making any statement of the type that because these sets are infinite they cannot be enumerated?"

For example, let's consider the number produced by stringing together the digits of any of the infinite subsets of the power set of the integers. This number (if it's a number) looks like an integer; it has no radix point in front of it. It consists of a string of digits in some base system. It's a string of digits just like any respectable integer, and it would be easy, (in principle of course) to find its predecessor; just subtract one! Is it an integer, a number or infinity ? If it's an integer then can the numbers from which it was constructed not be integers? Since all the problems of the alleged nonenumerability of the powerset comes from the completed infinity school which insists on having numbers assigned to infinite sets, it seems the fair question is to ask the finitist infinititists to first stick to their own principles and produce the infinite set to which they demand that a number be assigned. Is the concept of an infinite integer another sophistry designed to get around the contradictions introduced by the mixed mode use of infinity and the resulting sophistry of the nonenumerability of the power set while things (dare we say numbers !) even larger can be enumerated? Every integer is finite and of finite length, however somehow their "completed set" is infinite! And a nonfinite string of digits magically becomes not-a-number. This is all an attempt to produce the same kind of a "completed infinity" as in the case of the power series and integrals of curves of infinite length, but in this case, there is no magic bullet. The two views are very fundamentally and mutually incompatible.

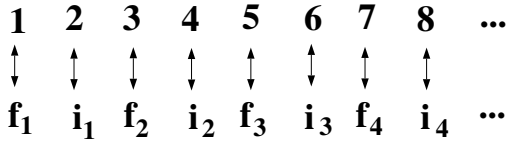
13. The Most Pedestrian and Convincing Enumeration of the Power Set

Finally, there is no reason as to why a simple tabular enumeration of the power set of the integers cannot be given. As before, if we indulge in the contemplation of the infinite number of subsets of

the power set with infinite number of members, we cannot imagine the point at which these subsets become infinite. We have the same problem as in the case of the even, odd or all the integers going to infinity and we don't know if it all happens at once. However from the viewpoint of the completed infinity school such sets obviously exist; for example the set of even integers, odd integers etc. Suppose then we assume that someone has listed them and given them to us. We can call these subsets S_1, S_2, S_3...; S_1 may be the odd integers, S_2 the even ones etc. We have an infinite number of them. We can then put them in a two dimensional table where the rows represent the sets and the columns represent the elements of the sets in some kind of an order as done for example in Boolos and Jeffrey, [p. 12, 1990. Please also see the appendix]. Since they are now all listed in this table, we can immediately enumerate them using the same kind of a method that Cantor used to enumerate the rationals. Obviously we can imagine a 1:1 mapping from this doubly infinite table to the doubly infinite table for the rationals produced by Cantor, of course, leaving out the integers which are only the first row in Cantor's enumerations of the rationals. (In fact it doesn't really matter, since Cantor's own transfinite arithmetic guarantees it.) Since by Cantor's own theorem that if **A** and **B** are enumerable sets, then their union is also enumerable, and since *at least the finite subsets* of the power set were enumerated in section 2, it follows immediately that the power set of the integers is enumerable.



However, since at least the the finite subsets were already shown to be enumerable(in fact all the sets can be enumerated with the process in section 2), we can also add this set, say S_0, to the list above and produce the final table, shown on the right. We could have added this set to the list anyway. The result is obviously analogous to Cantor's enumeration of the rationals. It should be noted that not only do we have enough integers to enumerate both the finite and infinite sets of the powerset, we even have *enough integers to enumerate each and every element of every infinite subset of the power set* of the integers. There is one possible (but invalid) objection to this tabular enumeration which is discussed in the section on transfinite arithmetic of Cantor. The very idea of a never-ending process (which is the inductive route to infinity) ever halting (or running out of integers in this case) before or after another never ending process is a blatant contradiction. The very definition of never halting is embedded firmly in the definition and the whole concept and process of enumeration! This result, of course, could have been obtained from Cantor's very own transfinite arithmetic; namely $\aleph_0 + \aleph_0 = \aleph_0$ which is merely a manifestation of his theorem that the union of two enumerable sets is enumerable. The final result is the same. All of the other previous enumerations were also valid. In fact if there is a problem with not being able to assign a single integer to any of the whole infinite subsets of the power set, we can duplicate the argument that Cantor himself used to show that if two sets were enumerable then their union was enumerable. This would only require a listing of the finite sets (f_i) and the infinite (i_i) sets next to one another and an enumeration process that jumps from one to the other as shown below. Once again, this was Cantor's proof that the union of two enumerable sets is enumerable.



14. Transfinite Arithmetic

There is yet one final objection that can be raised against the enumeration of the last section. One might ask how it would be possible to arrange the infinite number of infinite subsets of the integers in a neat array as shown in that section. What if the cardinality of the infinite subsets of the power set is not $\aleph_0$? If that were the case then the diagonalization proofs such as in Boolos & Jeffrey [please see appendix] would not be legal and no proof of that type would be legal since if we were really concerned with cardinality of this type (and since it's treated as a number) then the table would have a larger cardinality of rows then columns. There would really not be any diagonal in any sense and we'd have no idea what the antidiagonal set produced. Secondly, as in the case of the "proof" of the nonenumerability of the continuum, this argument produces a single set that is not in an infinite set. Lack of a single element cannot be used as a proof of smaller cardinality, since this would imply that a set such as the set of even integers which is missing the entire infinity of odd numbers could not have the same cardinality as all the integers. Furthermore, since at each step we only find a single set that's missing, we can just put this set in P(**Z**), and have the set complete. This is also the liar paradox in disguise. The only escape from this logic is to argue on the one hand that strange things happen with infinity (such as the equipollence of n, 2n and 2n+1, or even 3n) and then at the same time use the smallest contradiction to argue for nonequal cardinalities. The very definition of an infinite set as one which can be put into one-to-one correspondence with itself is a contradiction; and one can prove anything with a contradiction as a premise. Finally, if we can neatly line up an infinite number of sets, then we can line up the integers next to them. Of course, the lining up of the integers in a never-ending row is not the problem; we are forced to produce a function. The question feeds on itself again; how then do we produce the set of integers that are not divisible by seven? Again, it boils down to our inability produce a nice linear function from the integers to the powerset and has nothing to do with the intuitive understanding that the transfinite numbers (or the cardinalities of the infinite sets) are somehow proxies for "size" in some sense. It's the same application of using differing views of infinity in different places as is suitable for the theory.

However, the path taken in this section will be to show that Cantor's very own transfinite arithmetic is in contradiction with his proof of the nonenumerability of the power set of the integers. The first problem we confront when we try to imagine the infinite subsets of the power set is the need to account for an infinite number of elements permuted an infinite number at a time. We can use Cantor's definition: If we have a set **M** of *m* objects and a set **N** of *n* objects, the set m^n is the set which is the set of permutations of *m* objects *n* at a time where the repetetions of the *m* objects are permitted. The cardinal number of this number of permutations for infinite sets is defined by Cantor to be α^β where α is the cardinality of the set **M** and β the cardinal number of set **N**. Then since the cardinality of the integers is $\aleph_0$, this corresponds to the cardinal number $\aleph_0{}^{\aleph_0}$. This already contradicts the cardinality of the power set as being $2^{\aleph_0}$. If the Cantor symbolism is merely symbolism and means nothing other than the fact that $|P(\mathbf{Z})|$ greater than $\aleph_0$, then we can still try to compute what it is. If it really means that it's exponential in cardinality, which could be in line with our intuition that exponential time algorithms are fundamentally different than polynomial ones then it's less than what we might expect using Cantor's rule for permuting infinite sets.

The latter view is induction directly from finite sets; the former view is from the completed infinity view since it comes directly from the cardinal numbers of the sets. Which one is *more consistent*? We don't say *consistent*, since the very idea of a set being equal to a part of itself is inconsistent

from the point of view of classical logic and it shouldn't surprise anyone that proofs will produce lots of contradictions.

What is this cardinal number and can we compute it? We can compute it several different ways and show that they are all internally consistent. The first two identities we'll need are given by Cantor himself:

(13a) $\aleph_0 + \aleph_0 \equiv \aleph_0$

(13b) $\aleph_0 \cdot \aleph_0 \equiv \aleph_0$

The cardinal number $\aleph_0$ is idempotent under addition and multiplication. In fact (13b) is unnecessary, since using its definition in (13a) we can iterate it an $\aleph_0$ number of times to obtain;

(14) $\aleph_0 \equiv \aleph_0 + \aleph_0 = \aleph_0 + \aleph_0 + \aleph_0 = \aleph_0 + \aleph_0 + \aleph_0 + \ldots + = \aleph_0 \cdot \aleph_0$

If we now iterate (14) or equivalently (13a) we obtain

(15) $\aleph_0 \equiv \aleph_0 \cdot \aleph_0 = \aleph_0 \cdot \aleph_0 \cdot \aleph_0 = \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdot \ldots \cdot \aleph_0 = \aleph_0^{\aleph_0}$

This is identical to the result shown the earlier section when a single inductive infinity (∞) was used. Of course, there will objections to this result that the iteration is only legal for a finite number. Bu then there's no point to the protest since we can produce (13b) from (13a) via interation and it's consistent with Cantor's own production of it. The result (15) is the logical conclusion of a (cardinal) number which is indempotent. This result which is directly derivable from Cantor's cardinal numbers and his transfinite arithmetic is in direct contradiction of his proof that the cardinality of the power set is not enumerable but that the power set's cardinality is $2^{\aleph_0}$.

Furthermore (15) is also derivable from fundamental considerations which the master himself used. For example the mapping f:n→n! is implies that the factorials are also enumerable. This result itself should cause concern that the power set is not enumerable since by Stirling approximation;

(16) $\ln(n!) \approx n \cdot \ln(n) - n \approx n(\ln(n) - 1) \approx n \cdot \ln(n)$

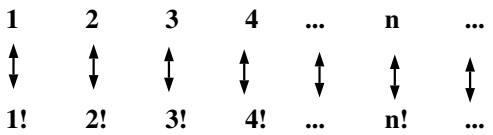
from which we know that

(17) $n! \approx n^n > 2^n$ for any $n > 2$

But since we don't want to be told again that strange things happen at infinity via enumeration, we can ignore it and use the definition of the factorial itself;

(18) $n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)\ldots$

Since we know from n → n! that n! is enumerable, that is, we can always pair the set as



then we have

(19) $$\aleph_0 = \aleph_0(\aleph_0-1)(\aleph_0 2)(\aleph_0-3)\ldots = \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \ldots = \aleph_0^{\aleph_0}$$

We obtain the same result using the enumeration process and of assigning cardinality due to this enumeration which is due to Cantor, and using transfinite arithmetic which is also due to Cantor. And once again, if the use of the identity for n! is illegal as an inductive tool for completed infinity and the transfinite numbers produced from such infinities (and which represent their *size* in some way), then what do we make of the cardinality of P(**Z**) as $2^{\aleph_0}$? Is it not a simple induction based on power sets of finite sets? If it is only a symbol, then its value is $\aleph_0$.

And these results including the results of the previous sections in which the power set is enumerated three different ways, and the result of the analysis of the Cantor proof for the nonenumerability of P(**Z**) which was shown to be a liar's paradox (--a contradiction in disguise buried in the proof very cleverly--) all point to the same conclusion; namely that the power set is not nonenumerable. The only thread by which the nonenumerability of the power set of the integers now hangs is that we cannot produce cute symbols which we can claim are "functions" which perform the mapping. However, if the transfinite arithmetic is the result of the enumeration processes and if it's to be used consistently, then the results of this arithmetic, despite the lack of symbolism, entail the enumerability of the power set, unless we can take refuge in the "strange mind-bending infinity" paradigm. But then, there are infinitely many different ways in which infinity can bend our minds.

15. Rational or Irrational?

Consider the number given by

(20a) = q+ 1 1+1! 1+2! 1+…+ 1…n!

or equivalently

(20b) = q+((n1)/n+((n1)/n)..+((n1)/n) {multiplied n times}

All of the terms are rational for all n. Since the product and sum of rationals is also rational, the number by construction-induction is rational. But we all know(!) that this number is irrational! We have another case of a number losing its property of being rational but the least number for which this happens is infinity. But if infinity is not a number then when does this number lose its property of being rational? There is no least number for which this number switches from being rational to being irrational. At the same time we also know that there exist rationals with infinite period which

also repeat (such a number was constructed earlier), hence having an infinite period is not equivalent to never repeating, therefore this number is either rational, irrational, both rational and irrational or if we don't believe that it can't be both and that there must be something wrong with the proofs, then it's neither rational nor irrational. The decision procedure by exhaustive enumeration tells us that we'll never be able to obtain the last digit of the first period so we can never tell if it repeats by this method. This brings up the old dispute between Kronecker of "God invented the integers" fame, and Cantor and abstract (axiomatic) set theorists. Which is more real; is the whole concept of a set a cheap abstraction of the concept of number, or is the concept of a set more fundamental and hence is a theory that subsumes number theories (and all others which are based on them) ? Has axiomatic set theory becom*e a glass bead* game?

Which shall we believe? Either the induction-construction is correct and the number is rational with an infinite period or some contradiction proof (which itself might be based on both induction and contradiction) is correct and the number is irrational. On the other hand, it's not clear that either induction or contradiction proofs themselves are free of contradictions since the idea of mapping the odd (or the even) integers to the whole set itself is not intuitively appealing to our sense of logic. Maybe the best thing to do is the accept the only logical conclusion; like the photon which is a wave and a particle which was grudgingly accepted by physicists, there are numbers which are both rational and irrational. In which case, we have at least two kinds of infinite set theories, constructible ones with their own intuition defying properties and imaginable-impredicative ones which contradict themselves and the constructivity-inductivity arguments. It doesn't seem to make any sense to accept infinite sets with contradictory properties and then use reductio ad absurdum proofs on them as if they were finite sets! The only thing that this whole concept says is that it is a theory of large sets which for all practical purposes may be considered to be infinite. This will bring it more in line with reality and other fields of mathematics such as using limits and shortcircuiting the problem. Of course, the imprecision which this generates might also have to be grudgingly accepted along with the problems it creates for predicate logic in which the symbol $(\forall x)$ occurs. But the very idea of defining sets by predicates itself as not being exactly precise also has to be accepted. Of course, we should only accept all of this provisionally!

16. Appendix A: Power Set Proofs

Four slightly different versions of *Cantor's Proof* --(one for each taste)--from the literature are given below with the sources delineated. Some changes in terminology and the symbols have been introduced to keep each of them as consistent with each other (regarding the symbols) as possible.

*The Cardinal of the Set of Subs*ets [Dieudunné, 1992]:

Let S be an arbitrary set. The mapping $s \rightarrow \{s\}$, which makes the subset of S --this is the element of P(S) whose only member is s -- correspond to each element $s \in S$, is a bijection from S onto a subset of P(S). We argue by contradiction from the assumption that there is such a mapping $f: S \rightarrow P(S)$. Let us consider the property $s \notin f(s)$ for element $s \in S$. By Zermelo's Axiom of Comprehension, there exists a subset $T \subset S$ formed of those $s \in S$ having this property. We show that $T \notin f(S)$, which contradicts the hypothesis that $f(S) = P(S)$. But if $s \in T$, we have by definition that $s \notin f(s)$, whence $f(s) \neq T$. On the other hand, if $s \notin T$, we have by definition that $s \in f(s)$, whence again $f(s) \neq T$. Thus T cannot be an element of f(S).

Neither the *Axiom of Infinity* nor the *Axiom of Choice* has been used.

*Proof of Cantor's Theorem [Devlin, 1*988]

Given an infinite set S, the proof demonstrates that P(S) does not have the same size as S (and hence is bigger). The idea is to show that the supposed existence of a matching between the elements of S and those of P(S) leads to a contradiction. If the members of S are denoted by letters of the alphabet (a finite set, of course, but good enough for this illustration), then part of such a supposed matching might look like this:

Element of S		Subset of P(S)
y	$\leftrightarrow$	{a,b,cd}= f(y)
z	$\leftrightarrow$	{b,d,p,,z}=f(z)

In this case y is not a member of the set f(y) with which it is matched, whereas z is a member of its matched set f(z). Let T denote the set of all those members of S which are not members of their matched set:

$$T = \{x \mid x \notin f(x)\}$$

Since T is a subset of S, T must be matched with some element of S, say s*:

$$s^* \leftrightarrow T = f(s^*)$$

Investigation of whether or not s* is a member of the set f(s*) now leads to a contradiction. If s* is a member of f(s*), then this means that s* cannot be in T, but since T is just f(s*), then s* will be a member of T, and again since T = f(s*), there is a contradiction.

*Cantor's Proof That a Power Set Has More Elements than the Set Generating It [Maor, 1*987]

Let S be the set of integers and S´ be its power set. Assume that a 1:1 correspondence exists. This means that for every $s \in S$ there exists one, and only one, element $s' \in S'$. Let us call the element s whose image is s´ *the gener*ator of s´. Now the elements of S´ can be classified into two classes: those elements (i.e. subsets of S´) which contain their own generators, and those which don't. We now form the set T consisting of all those elements of S whose images belong to the second class in S´; that is, all the elements of S which are not contained in their own images. Since T consists of elements of S, it is itself a subset of S, and therefore is an element of S´. And since there exists a 1:1 correspondence between S and S´, T must have a generator in S. Call this generator s*.

We now ask: Is s* contained in T? If it is, this would contradict the definition of T, as the set of all elements of S which are not contained in their own images. So s* is not contained in T. But this again contradicts the definition of T, since T is the set of all elements of S which are not contained in their images. Thus s* at the same time must and must not be contained in T, which is obviously impossible. Therefore the assumption that a 1:1 correspondence between S and S´ can be established as false. Thus we have established that a set and its power set must have different cardinalities. To establish which one has a greater cardinality, we note that each element $s \in S$ can always be matched with one element of S´ through the trivial correspondence $s \rightarrow \{s\}$. Thus every element in S has an image in S´, but not every element in S´ has a generator in S. Therefore S´ has more elements than S: The cardinality of a power set is always greater than the cardinality of its "parent" set.

*Diagonalization [Nonenumerability of the Power Set] [Boolos & Jeffrey, 1*991]

Confronted with any infinite list

$$S_1, S_2, S_3, \ldots$$

of sets of positive integers, we define a set D(L) as follows:

(*) *For each positive integer n, n is in D(L) if and only if n is not in* S_n.

The statement (*) genuinely defines a set D(L) for, given any positive integer, n, we can tell whether n is in D(L) if we can tell whether n is in *t*he nth set in list L. We argue by reductio ad absurdum: we suppose that D(L) does appear somewhere in list L, say as entry number m, and deduce a contradiction, thus showing that the supposition must be false. Assume that for some positive integer m, we have $S_m = D(L)$. To deduce a contradiction we apply the definition (*) to the positive integer m: with n=m, the statement (*) tells us that;

m is in D(L) if and only if m is not in Sm.

Now a contradiction follows from our supposition: if S_m and D(L) are the one and same set we have:

m is in D(L) if and only if m is not in D(L).

Since this is a flat self-contradiction, our supposition must be false. For no positive integer m do we have S_m=D(L). In other words, the set D(L) is named nowhere in the list. Applied to any list of sets of positive integers it yields a set of positive integers which was not in the list. There is no list that enumerates all sets of positive integers: the set P(S) of all such sets is not enumerable.

The method (diagonalization) can be looked at from a slightly different point of view. We can think of the $S_1,S_2,S_3,...$ as represented by functions $s_1,s_2,s_3,...$ of positive integers which take the numbers 0 and 1 as values. The relationship between the set S_n and the corresponding function s_n is simply this: for each positive integer p we have:

$$s_n(p) = \begin{array}{ll} 1 & \text{if p is in } S_n \\ 0 & \text{if p is not in } S_n \end{array}$$

Then the list L can be visualized as an infinite rectangular array of zeros and ones as in the table

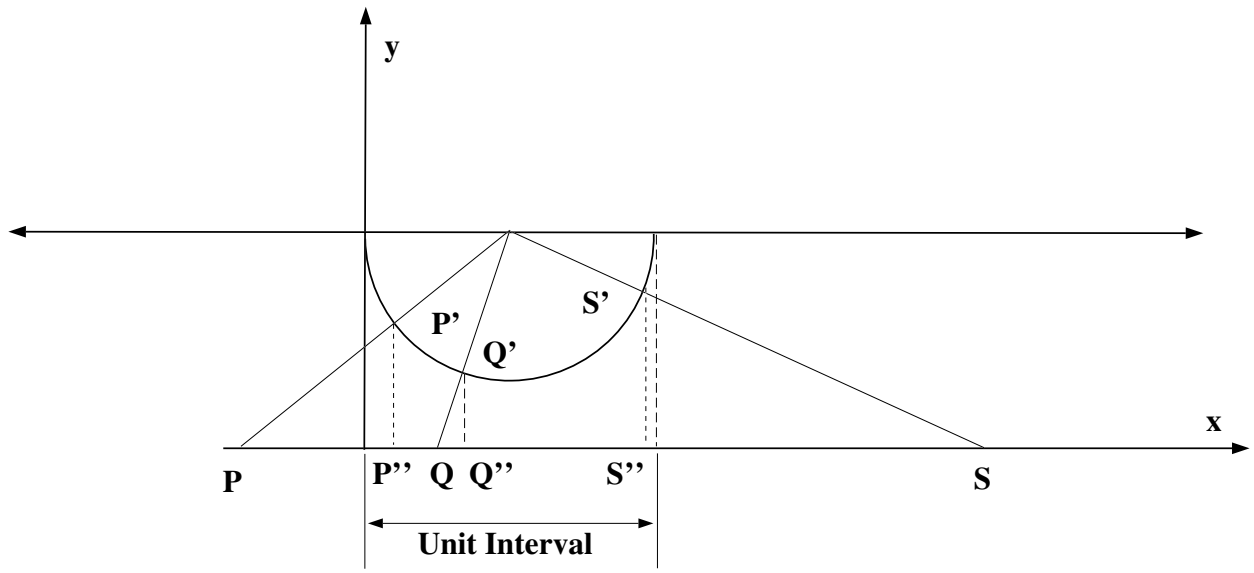
	1	2	3	...
s_1	$s_1(1)$	$s_1(2)$	$s_1(3)$	...
s_2	$s_2(1)$	$s_2(2)$	$s_2(3)$	...
s_3	$s_3(1)$	$s_3(2)$	$s_3(3)$	...
...	...	...	...	...

The nth row represents the function s_n and hence the set S_n. The main diagonal given by $s_1(1),s_2(2),s_3(3),..s_n(n),...$ forms a sequence of zeros and ones. There might be an integer d such that the set S_d is the diagonal set. However we can always generate a set, the diagonal set, which will always guarantee producing a set which is nowhere in the list L. This is the antidiagonal set which consists of the set which we obtain by flipping the bits in the main diagonal. This antidiagonal sequence differs from every row of the array and hence differs from every set in the list L.

*[It should be noted that we have found only a single set that doesn't belong to an infinite set. This is similar to the proof of the nonenumerability of the continuum and hence the liar paradox. Infinite sets, which are subsets of infinite sets can have an infinite number of elements missing and yet still have the same cardinality as the whole infinite set. To conclude that an infinite set has a smaller cardinality than some other set because a single element is missing is an absurdity, and is nothing but the finite set definition of an infinite set which is implemented with reductio ad absurdum proof*s].

Appendix B: Cantor's Proof that the Real Numbers Have the Cardinal Number c

A figure showing that there's a one-to-one correspondance between the point set (0,1) and the real number line. A point P´´ in the interval (0,1) can be matched with a point P´ on the semicircle and then this point is matched with a point P on the real number line. The procedure can also be carried out in reverse.



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